HIGH-ORDER PSEUDO-ANALYTICAL METHOD FOR ACOUSTIC WAVE MODELING

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ABSTRACT

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For the time evolution of acoustic wavefields we present an alternative derivation of the pseudo-analytical method, which enables us to generalize the method to high-order formulations. Within the same derivation framework, we compare the second-order pseudo-analytical method, the Fourier finite difference method, and the fourth-order Lax-Wendroff time integration method. We demonstrate that the pseudo-analytical method can be regarded as a modified Lax-Wendroff method. Different from the fourth-order time stepping method, both the second-order pseudo-analytical method and the Fourier finite difference method use pseudo-Laplacians to compensate for time stepping errors. The pseudo-Laplacians need to be solved in the wavenumber domain with constant compensation velocities for computational simplicity and efficiency. Low-order pseudo-Laplacians are more sensitive to the choice of compensation velocities than high-order ones. As a result, we need to use the combination of several pseudo-Laplacians to achieve the required accuracy for low-order pseudo-analytical methods. When using the pseudospectral method to evaluate all spatial derivatives, the computation cost for the second-order pseudo-analytical method, the Fourier finite difference method, and the fourth-order Lax-Wendroff time integration method is approximately the same. Both the second-order pseudo-analytical method and the Fourier finite difference method have less restrictive stability conditions than the fourth-order time stepping method. We demonstrate with numerical examples that the second-order pseudo-analytical method, greatly improves the original pseudo-analytical method and as a modified version of the Lax-Wendroff method, is well suited for imaging seismic data in subsalt areas where reverse-time migration plays a crucial role.

KEY WORDS: acoustic wave equation, seismic modeling, pseudo-method, pseudo-analytical method, pseudo-Lapliacan operator, Fourier finite-difference method, Lax-Wendroff method, Fourier pseudo-method.

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INTRODUCTION

Seismic wave simulation remains an important branch of exploration seismology, functioning as the backbone of many problems including seismic modeling, imaging and full waveform inversion, to name a few. The approach of directly seeking solutions to wave equations using numerical methods becomes more and more appealing with the ever increasing computing power that is now widely available. To solve time-domain wave equations on discrete grids, both the spatial derivatives and the temporal derivative need to be discretized. Many numerical methods have been applied to tackling spatial derivative discretizations. Popular choices include the finite difference method, the pseudospectral method, the finite element method, and their variants. Though alternative options are available, the dominant method of choice for temporal derivative discretizations has been the finite difference method, especially, the second-order finite difference method. It has been recognized that temporal discretizations can introduce noticeable numerical errors even if highly accurate spatial operators are adopted. The conventional way of solving this problem is to use high-order time stepping schemes, or use optimized methods (Dablain, 1986; Crase, 1990; Ghrist et al., 2000; Chen, 2006, 2007; Zhang et al., 2007; Soubaras and Zhang, 2008; Chu et al., 2009).

The methodology of most high-order time stepping schemes is not very different from that of high-order spatial operators. In either case, the high accuracy is achieved by using high-order polynomial or rational approximations to the theoretical analytical expressions that usually have no closed forms (Chu, 2009). It is possible to directly compute the non-closed form analytical formulas without having to expand them and taking different degrees of truncations. Two of such methods have been proposed by Kole (2003) and Zhang and Zhang (2009). Both of these methods have explicit formulations similar to the conventional explicit time stepping schemes but they are unconditionally stable. This indeed is very attractive but it unfortunately comes with a price. The matrix exponential approach (Kole, 2003) requires algorithms to compute exponentials of matrices. Memory and speed efficiency become an issue for realistic size models. Similar to the matrix exponential method, the one-step extrapolation method (Zhang and Zhang, 2009) also has a pseudo-differential term to deal with, expressed as a square root function rather than exponential. Pestana and Stoffa (2010) reintroduced the Rapid Expansion method of Kosloff et al. (1989) and Tal-Ezer et al. (1987). Here the exponential of the pseudo-differential operator is expanded in terms of orthogonal Chebyshev polynomials. This method is particularly attractive when high accuracy results are required and for large time steps.

The pseudo-analytical method attempts to solve the problem in a different way (Chu,2009; Etgen and Brandsberg-Dahl, 2009). Rather than seeking accurate approximations to the analytical expressions, the pseudo-analytical method employs modified spatial derivatives to compensate for the errors caused by the second-order time stepping scheme. The modified spatial derivatives form a pseudo-differential operator that requires special treatment, similar to the matrix exponential method and the one-step extrapolation method. To compute this pseudo-differential operator efficiently, the pseudo-analytical method simplifies it by assuming a constant velocity medium which results in a formula that can be easily calculated in the wavenumber domain. It then relies on the combination of several such pseudo-differential operators each for a different constant velocity to accommodate actual velocity variations. Obviously, the pseudo-analytical method becomes expensive for complex models with high velocity contrasts since we will need to compute more pseudo-Laplacians. The Fourier finite difference method (Song et al., 2010), though proposed in a different theoretical framework, takes a very similar form as the pseudo-analytical method. The Fourier finite difference method divides the wavefield computation into two steps. The first step is to compute the pseudo-Laplacian in the same way that the pseudo-analytical method does. The second step is to apply a finite difference operator to the output wavefield from the first step. This two-step approach partially alleviates the problem of being sensitive to velocity variations. When using the second-order operator to conduct the finite difference computations, as was proposed by Song et al. (2010), the Fourier finite difference method only slightly increases the computation cost.

In this paper, we give an alternative derivation for the pseudo-analytical method which leads to high-order pseudo-analytical methods. We first derive the pseudo-analytical method with Taylor series expansions and show that high-order pseudo-Laplacians are far less sensitive to the compensation velocity than low-order ones. We then compare the second-order pseudo-analytical method with the Fourier finite difference method and the fourth-order Lax-Wendroff time integration method. All these three methods involve two computational steps. We show that the basic idea behind the second-order pseudo-analytical method and the Fourier finite difference method is the same and that the pseudo-analytical method can be regarded as a modified Lax-Wendroff method.

HIGH-ORDER PSEUDO-ANALYTICAL METHOD

Consider the constant-density acoustic wave equation

$$(1/v^2)(\partial^2P/\partial t^2) = (\partial^2P/\partial x^2) + (\partial^2P/\partial y^2) + (\partial^2P/\partial z^2) \quad , \tag{1}$$

where $P = P(x,y,z,t)$ represents pressure and is a function of position and time and v is the P-wave velocity. Assume v is constant for the moment. Performing a spatial 3D Fourier transform gives

$$\partial^2\bar{P}/\partial t^2 = -v^2(k_x^2 + k_y^2 + k_z^2)\bar{P} = -v^2|\mathbf{k}|^2\bar{P} \ , \tag{2}$$

where $\bar{P} = P(k_x,k_y,k_z,t)$ stands for the 3D spatial Fourier transform of P(x,y,z,t) and k_x, k_y, k_z are wavenumbers. The analytical solution, if we know the initial values $\bar{P}_0$ and $\partial\bar{P}_0/\partial t$ at t = 0 (Etgen, 1989), is

$$\bar{P} = \cos(v|\mathbf{k}|t)\bar{P}_0 + [\sin(v|\mathbf{k}|t)/v|\mathbf{k}|](\partial\bar{P}_0/\partial t) \ . \tag{3}$$

If a three-level time stepping scheme is adopted, eq. (3) can be simplified to

$$\bar{P}^{n+1} + \bar{P}^{n-1} = 2\cos(v\Delta t|\mathbf{k}|)\bar{P}^n \ , \tag{4}$$

where $\bar{P}^n = \bar{P}(n\Delta t)$. The pseudo-analytical method proposed by Etgen and Brandsberg-Dahl (2009) is now derived from (4) by adding $-2\bar{P}^n$ to both sides

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = 2[\cos(v\Delta t|\mathbf{k}|) - 1]\bar{P}^n \ , \tag{5}$$

and rewriting it into the following form

$$(\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1})/\Delta t^2 = v^2\{2[\cos(v\Delta t|\mathbf{k}|) - 1]/(v\Delta t)^2\}\bar{P}^n \ , \tag{6}$$

If we approximate the second order time derivative in (2) by a 2-nd order finite-difference scheme and compare it with (6), we have the pseudo-Laplacian operator which is defined as

$$F(\mathbf{k}) = 2[\cos(v\Delta t|\mathbf{k}|) - 1]/v_0^2\Delta t^2 \approx - |\mathbf{k}|^2 + (v_0^2\Delta t^2/12)|\mathbf{k}|^4 - \cdots \ . \tag{7}$$

The Taylor expansion of the operator cancels the dependence of the velocity in the first term and this will increase the interpolation accuracy for small times steps. F(**k**) is only exactly the Fourier transform of the Laplacian operator $-|\mathbf{k}|^2$ in the limit as the time step size approaches zero.

Thus the pseudo-analytical method introduced by Etgen and Brandsberg-Dahl (2009) is given by:

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = (v\Delta t)^2F(\mathbf{k})\bar{P}^n \ . \tag{8}$$

v_0 in eq. (7) is the compensation velocity, which is a constant for each pseudo-Laplacian. Because the pseudo-Laplacian F(**k**) only slowly varies with v_0, we may use the combination of several pseudo-Laplacians to better accommodate velocity variations.

Classical finite-difference schemes are derived in terms of a Taylor series. In this way, replacing the second derivative in time in eq. (2) by its Taylor series representation, we can rewrite (2) as:

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = -v^2\Delta t^2|\mathbf{k}|^2\bar{P}^n + O(v\Delta t|\mathbf{k}|)\bar{P}^n \ , \tag{9}$$

where the term $O(v\Delta t|\mathbf{k}|)$ involves the others derivatives of $\bar{P}^n$ in time.

Substituting (4) into eq. (9), it follows that

$$O(v\Delta t|\mathbf{k}|) = 2[\cos(v\Delta t|\mathbf{k}|) - 1 + v^2\Delta t^2|\mathbf{k}|^2/2] \ . \tag{10}$$

With the value of $O(v\Delta t|\mathbf{k}|)$ evaluated in eq. (10), we can now derive from eq. (9) a second-order pseudo-analytical method that is given by

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = -(v\Delta t|\mathbf{k}|)^2\bar{P}^n + (v\Delta t)^4F_2(\mathbf{k})\bar{P}^n \ , \tag{11}$$

where

$$F_2(\mathbf{k}) = 2[\cos(v_0\Delta t|\mathbf{k}|) - 1 + \tfrac{1}{2}(v_0\Delta t|\mathbf{k}|)^2]/(v_0\Delta t)^4 \ , \tag{12}$$

is the second-order pseudo-Laplacian.

For clarification, we now call $F(\mathbf{k})$ given in eq. (7) the zero-th order pseudo-Laplacian and the original pseudo-analytical method given in (8) the zero-th order pseudo-analytical method. The first half on the right hand side of eq. (11) is just the normal pseudospectral method, with the second-order finite difference method applied to the temporal derivative. The second half on the right hand side of eq. (11) acts like a correction term, which makes (11) very similar to the fourth-order Lax-Wendroff scheme. In fact, eq. (11) can be regarded as an improved version of the fourth-order Lax-Wendroff time stepping method, as we will prove in a later section. This alternative interpretation of the pseudo-analytical method leads to a simple way of generalizing it to high-order formulations.

The corresponding time domain equation for (11) is

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = (v\Delta t)^2\nabla^2\bar{P}^n + (v\Delta t)^4\mathrm{IFFT}\{F_2(\mathbf{k})\bar{P}^n\} \ , \tag{13}$$

where IFFT stands for spatial inverse Fourier transform and $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$.

Following the same procedures, we can derive the fourth-order pseudo-analytical method for eq. (1) straightforwardly

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = -(v\Delta t)^2|\mathbf{k}|^2\bar{P}^n + (1/12)(v\Delta t)^4|\mathbf{k}|^4\bar{P}^n + (v\Delta t)^6F_4(\mathbf{k})\bar{P}^n \ , \tag{14}$$

where

$$F_4(\mathbf{k}) = 2[\cos(v_0\Delta t|\mathbf{k}|) - 1 + ½(v_0\Delta t|\mathbf{k}|)^2$$
$$- (1/24)(v_0\Delta t|\mathbf{k}|)^4]/(v_0\Delta t)^6 \ , \quad (15)$$

is the fourth-order pseudo-Laplacian. Other high-order formulas can be readily derived in a similar way.

Like the zero-th order pseudo-analytical method, the high-order pseudo-analytical method works for heterogeneous media because the high-order pseudo-Laplacians only vary slowly with the compensation velocity, v_0 [Figs. 1(c) to 1(f)]. Notice that all the pseudo-Laplacians have similar shape in the wavenumber domain. Better than the original pseudo-Laplacian [Figs. 1(a) and 1(b)], however, the high-order pseudo-Laplacians are far less sensitive to velocity variations. As can be observed in Fig. 1: $F(\mathbf{k})$ remains approximately constant only for the low wavenumbers; $F_2(\mathbf{k})$ covers a wider nearly-constant range than $F(\mathbf{k})$; $F_4(\mathbf{k})$ only noticeably varies for very high wavenumbers. In general, the magnitude of the pseudo-Laplacians is inversely proportional to the order. As a result, we may choose to use a single compensation velocity to compute the pseudo-Laplacians without having to do interpolations or smoothing for high-order pseudo-analytical methods. Therefore, high-order pseudo-analytical methods are not only more accurate but also potentially more efficient.

COMPARISON WITH THE FOURIER FINITE DIFFERENCE METHOD

The Fourier finite difference method can be derived from (5) but it tackles the problem in a slightly different way. Rather than writing eq. (5) in the form of eq. (6), the Fourier finite difference method introduces the compensation velocity v_0 as follows (Song et al., 2010)

$$\bar{P}^{n+1} - 2\bar{P}^n + \bar{P}^{n-1} = 2[\cos(v\Delta t)|\mathbf{k}| - 1]$$
$$\times \{[\cos(v\Delta t)|\mathbf{k}| - 1]/[\cos(v_0\Delta t)|\mathbf{k}| - 1]\}\bar{P}^n \ . \quad (16)$$

With the following equation, which is derived based on the first-order Taylor series expansion at

$$[\cos(v\Delta t)|\mathbf{k}| - 1]/[\cos(v_0\Delta t)|\mathbf{k}| - 1]$$
$$\approx (v^2/v_0^2) - [v^2(v^2 - v_0^2)/12v_0^2]\Delta t^2|\mathbf{k}|^2 \ , \quad (17)$$

eq. (16) now becomes

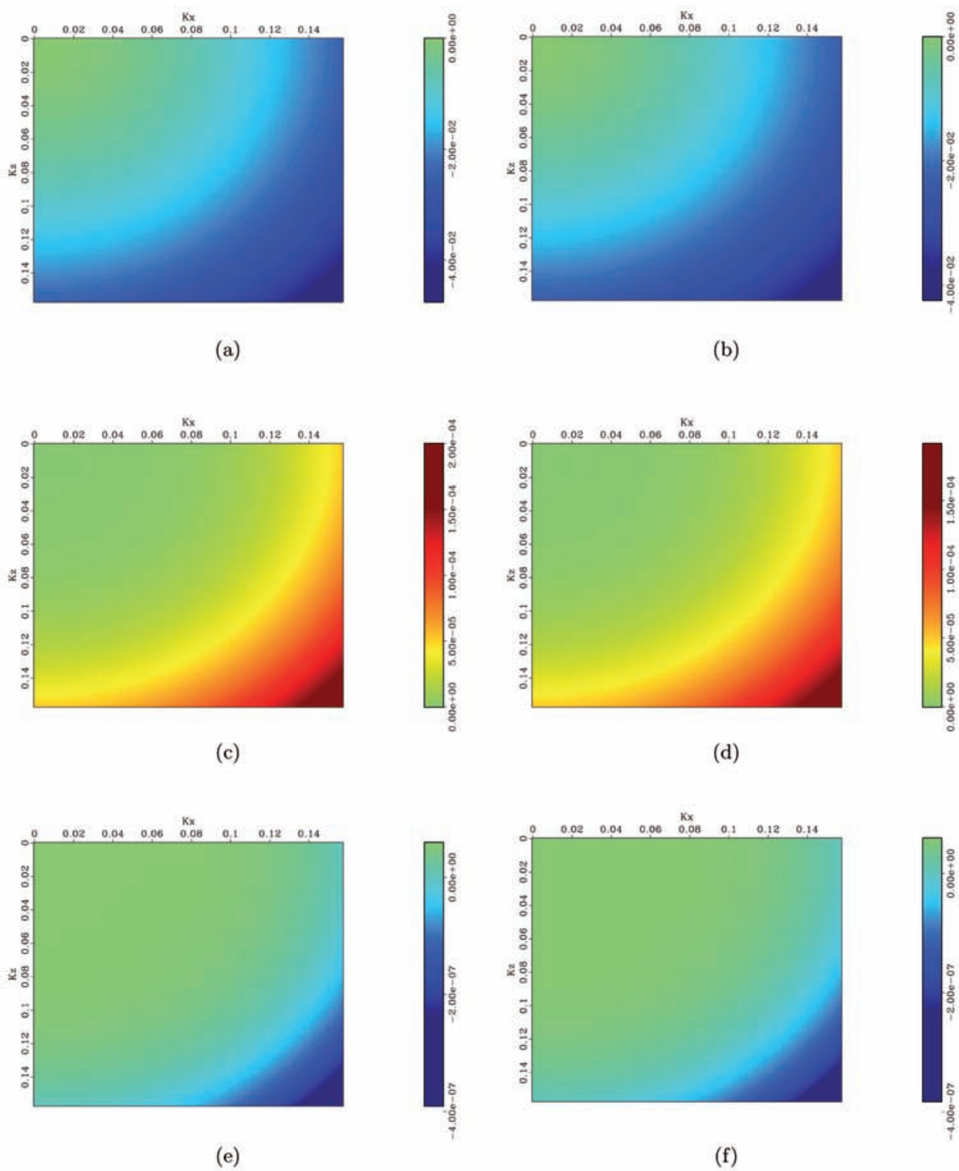

Fig. 1. Pseudo-Laplacian for $v_0 = 1500$ m/s. (b) Pseudo-Laplacian for $v_0 = 3000$ m/s. (c) Second-order pseudo-Laplacian for $v_0 = 1500$ m/s. (d) Second-order pseudo-Laplacian for $v_0 = 3000$ m/s. (e) Fourth-order pseudo-Laplacian for $v_0 = 1500$ m/s. (f) Fourth-order pseudo-Laplacian for $v_0 = 3000$ m/s.

$$\bar{P}^{n+1} - 2\bar{P}^{n} + \bar{P}^{n-1} = \{(v\Delta t)^2 - [v^2(v^2 - v_0^2)/12]\Delta t^4|\mathbf{k}|\}[F(\mathbf{k})\bar{P}^n] \ . \quad (18)$$

Similar to the pseudo-analytical method, the pseudo-Laplacian F(**k**) needs to be computed in the wavenumber domain. The other wavenumber term on the right hand side of (18) can be computed either in the space domain using finite differences or in the wavenumber domain using the pseudospectral method. This becomes more obvious after we write eq. (18) in its corresponding time domain form

$$\bar{P}^{n+1} - 2\bar{P}^{n} + \bar{P}^{n-1} = (v\Delta t)^2 Q_{ffd}^n + [v^2(v^2 - v_0^2)\Delta t^4/12]\nabla^2 Q_{ffd}^n \ , \quad (19)$$

$$Q_{ffd}^n = \mathrm{IFFT}\{F(\mathbf{k})\bar{P}^n\} \ . \quad (20)$$

Song et al. (2010) proposed to use the second-order finite difference scheme to evaluate ∇^2 and called it the FFD method. Obviously, the operator ∇^2 can also be computed using the pseudospectral method. We call this approach the FPS method.

Both the Fourier finite difference method and the second-order pseudo-analytical method contain a wavenumber term with a constant compensation velocity v_0. The difference between these two methods is that the Fourier finite difference method chooses to evaluate the wavenumber term first and then conduct spatial differentiations, while the second-order pseudo-analytical method conducts the spatial differentiations first followed by the wavenumber domain computations. High-order Fourier finite difference methods can be straightforwardly derived using higher-order terms in the Taylor series. For instance, the second-order Taylor series expansion around $|\mathbf{k}|^2 = 0$ can be found as

$$[\cos(v\Delta t)|\mathbf{k}| - 1]/[\cos(v_0\Delta t)|\mathbf{k}| - 1]$$
$$\approx (v^2/v_0^2) - [v^2(v^2 - v_0^2)/12v_0^2]\Delta t^2|\mathbf{k}|^2$$
$$+ \{[2v^2(v^4 - v_0^4) - 5v^2v_0^2(v^2 - v_0^2)]/360v_0^2\}\Delta t^4|\mathbf{k}|^4 \ . \quad (21)$$

The corresponding Fourier finite difference formula therefore is

$$\bar{P}^{n+1} - 2\bar{P}^{n} + \bar{P}^{n-1}$$
$$= \{(v\Delta t)^2 - [v^2(v^2 - v_0^2)/12]\Delta t^4|\mathbf{k}|^2\}[F(\mathbf{k})\bar{P}^n]$$
$$+ \{[2v^2(v^4 - v_0^4) - 5v^2v_0^2(v^2 - v_0^2)]/360\}\Delta t^6|\mathbf{k}|^4[F(\mathbf{k})\bar{P}^n] \ . \quad (22)$$

COMPARISON WITH THE LAX-WENDORFF METHOD

We now show that the high-order pseudo-analytical method is simply a variant of the Lax-Wendroff scheme. Consider the second-order pseudo-analytical method, eq. (11) for example. If we approximate the cosine function in the second-order pseudo-Laplacian with the fourth-order Taylor series, we derive

$$\bar{P}^{n+1} - 2\bar{P}^{n} + \bar{P}^{n-1}$$

$$= -(v\Delta t|\mathbf{k}|)^2\bar{P}^n - (1/12)(v\Delta t)^2|\mathbf{k}|^2[-(v\Delta t)^2|\mathbf{k}|^2\bar{P}^n] \ , \quad (23)$$

which is the fourth-order Lax-Wendroff time stepping scheme. In the time domain, eq. (23) reads

$$\bar{P}^{n+1} - 2\bar{P}^{n} + \bar{P}^{n-1} = Q^n + (1/12)(v\Delta t)^2\nabla^2 Q^n \ , \quad (24)$$

$$Q^n = (v\Delta t)^2\nabla^2\bar{P}^n \ . \quad (25)$$

Similarly, we may derive the sixth-order Lax-Wendroff scheme by expanding the cosine function in (15) with the sixth-order Taylor series.

The above derivation provides new insights into the pseudo-analytical method. With this derivation, we may now interpret the pseudo-analytical method as a modified Lax-Wendroff scheme. The Lax-Wendroff method achieves high accuracy by using high order truncations to the cosine function. The pseudo-analytical method achieves the same goal of obtaining high accuracy by keeping the remainder terms and evaluating these remainder terms approximately in the wavenumber domain. The truncation approach of the Lax-Wendroff scheme results in explicit differential operators that can be easily discretized by all commonly used numerical methods. On the contrary, the pseudo-analytical method is restricted to the wavenumber domain because the remainder terms are pseudo-differential operators that are more convenient to be solved numerically in that domain for homogeneous media. Another implication of the above derivations is that other types of polynomials might be used instead of the Taylor series expansions to derive the pseudo-analytical method that might lead to better accuracy and/or efficiency.

STABILITY CRITERION

We now consider the problem of establishing a criteria for stability. To make the numerical computation stable the time interval has to be small enough to satisfy the stability condition.

First we consider the pseudo-analytical method, eq. (8). Canceling $\bar{P}^{n-1}$ leaves a quadratic equation for $\bar{P}$:

$$\bar{P}^2 - 2\bar{P} + 1 = (v\Delta t)^2 F(\mathbf{k})\bar{P} \ . \tag{26}$$

This equation has two solutions for $\bar{P}$. For stability, both must satisfy $|\bar{P}| \leq 1$. Rewrite eq. (26) for $\bar{P}$:

$$\bar{P}^2 - 2[1 + ½(v\Delta t)^2 F(\mathbf{k})]\bar{P} + 1 = 0 \ . \tag{27}$$

The roots of $\bar{P}^2 - 2a\bar{P} + 1 = 0$ are $\bar{P} = a \pm \sqrt{(a^2 - 1)}$. Everything depends on the square root giving an imaginary number. For $a^2 = [1 + ½(v\Delta t)^2 F(\mathbf{k})]^2 \leq 1$: if $a^2 \leq 1$ then $\bar{P} = a \pm i\sqrt{(1 - a^2)}$ has $|\bar{P}|^2 = a^2 + (1-a^2) = 1$.

The condition $½(v\Delta t)^2 F(\mathbf{k}) < 0$ does produce $a^2 \leq 1$ and this method will be stable.

Thus,

$$½(v\Delta t)^2 F(\mathbf{k}) = v^2[\cos(v_0 \Delta t|\mathbf{k}|) - 1]/v_0^2 < 0 \ . \tag{28}$$

To satisfy the equation above implies that $|\cos(\phi)| < 1$ where $\phi = v_0 \Delta t \sqrt{(k_x^2 + k_y^2 + k_z^2)}$. For the maximum frequency present in the data we have:

$$\phi = v_0 \Delta t \sqrt{(k_x^2 + k_y^2 + k_z^2)} = 2\pi \Delta t f \leq 2\pi \Delta t f_{max} \ . \tag{29}$$

The maximum Δt we can use that still ensures accuracy for this method is a time sampling based on the maximum frequency present in the data. In this way, $f_{max} = 1/(2\Delta t)$, which corresponds to $\phi = \pi$.

Considering a 2D case, taking the highest spatial frequencies $k_x = \pi/\Delta x$ and $k_z = \pi/\Delta z$ and for $\Delta x = \Delta z$, from eq. (29) we obtain

$$\alpha = (\Delta t v_0/\Delta x) \leq \sqrt{2}/2 \ . \tag{30}$$

Using the same procedure, we can extend this approach to the higher order pseudo-analytical methods. For the second-order pseudo-analytical method, eq. (11), we have that

$$½[-(v\Delta t|\mathbf{k}|)^2 + (v\Delta t)^4 F_2(\mathbf{k})] < 0 \ , \tag{31}$$

which will produce $a^2 \leq 1$.

Now, if we rewrite the operator $F_2(\mathbf{k})$ as

$$F_2(\mathbf{k}) = [F_2(\mathbf{k}) + |\mathbf{k}|^2]/(v_0\Delta t)^2 \ ,$$

and substitute into (31), we have that

$$[(v\Delta t|\mathbf{k}|)^2/2]\{[(v^2/v_0^2) - 1] + v^2F(\mathbf{k})/v_0^2|\mathbf{k}|)^2\} < 0 \ . \tag{32}$$

We take the following approximation for $F(\mathbf{k})$, i.e, $F(\mathbf{k}) = -|\mathbf{k}|^2 + (v_0^2\Delta t^2/12)|\mathbf{k}|^4$ and considering again just the 2D case, with $k_x = \pi/\Delta x$ and $k_z = \pi/\Delta z$ and for $\Delta x = \Delta z$, we have that

$$\Delta t < \sqrt{6}\,\Delta x/\pi v \ . \tag{33}$$

Thus the time-stepping for the second order pseudo-analytical, using only the second order approximation for $F(\mathbf{k})$, has to satisfy (33) to be stable.

For the FPS method, given by eq. (18), using the same procedure, it will be also stable if,

$$\tfrac{1}{2}\{(v\Delta t)^2 - [v^2(v^2 - v_0^2)/12]\Delta t^4|\mathbf{k}|^2\}F(\mathbf{k}) < 0 \ . \tag{34}$$

If we consider that $F(\mathbf{k}) < 0$ this implies that $(v_0\Delta t|\mathbf{k}|)$ has to be less then 1. Thus, we have that

$$\Delta t \leq \sqrt{2}\,\Delta x/2v_0 \ . \tag{35}$$

But the first term on the left of 34 has to be

$$(v\Delta t)^2\{1 - [v^2(v^2 - v_0^2)/12]\Delta t^2|\mathbf{k}|^2\} > 0 \ . \tag{36}$$

Then,

$$[(v^2 - v_0^2)/12]\Delta t^2|\mathbf{k}|^2\} < 1 \ , \tag{37}$$

and

$$\Delta t < \sqrt{6}\,\Delta x/\pi\sqrt{(v^2 - v_0^2)} \ . \tag{38}$$

We considered here the same 2D case as before, but the FPS had to satisfy simultaneously eqs. (35) and (38) to be stable. However, in the case where $v = v_0$, the FPS method reduces to the first order pseudo-analytical method and both satisfy the same stability condition which is given by eq. (30).

NUMERICAL EXAMPLES AND DISCUSSION

First we use a two-layer model to show a series of numerical examples and compare the results for the methods considered. The model has an upper layer with v = 1500 m/s and a lower model with v = 4500 m/s. The grid spacing on this model is 15 m for both the horizontal and vertical directions. In the numerical examples, we use a Ricker source wavelet with a maximum frequency equal to 50 Hz located in the upper layer. Each image shows a wavefield snapshot at t = 1.0 s. Fig. 2 shows computation using the pseudo-analytical method for time-step values of $\Delta t = 0.001$ s, 0.0015 s, and 0.0025 s using the compensation velocity of $v_0 = 1500$ m/s on the left and $v_0 = 4500$ m/s on the right. Based on the stability study, which considered a homogeneous medium, we found that the pseudo-analytical method, relation (30), requires $\Delta t < \sqrt{2}\,\Delta x/2v_0$ in order to be stable. Taking the maximum velocity in the medium, i.e, $v_0 = 4500$ m/s, and for $\Delta x = \Delta z = 15$ m, we find that the time step has to be less than 0.0023 s. For this example, we find that for time steps of $\Delta t = 0.001$ s and $\Delta t = 0.0015$ s, and taking the compensation velocity equal to the minimum velocity results in data that are free of oscillatory noise. The results for $\Delta t = 0.001$ s are also free of noise for both maximum and minimum velocity compensation. However, for $\Delta t = 0.0015$ s and $\Delta t = 0.0025$ s, the results display oscillatory artifacts when the compensation velocity is equal to the maximum velocity. Thus, we confirm that the influence of v_0 used in the pseudo-Laplacian operator is very small, but only for small time steps, which we can see by comparing the results of Figs. 2(a) and 2(b) and Figs. 2(c) and 2(d).

In Fig. 3 we show the results for time steps $\Delta t = 0.0025$ s for the second-order pseudo-analytical method and FPS, using the compensation velocity $v_0 = 1500$ m/s on the left and $v_0 = 4500$ m/s on the right, and the fourth-order Lax-Wendroff. We used the FPS method instead of FFD method to eliminate any errors that could be associated with the finite-difference operator. The results with the second order pseudo-analytical and Lax-Wendroff methods, Figs. 3(a), 3(b) and 3(e), respectively, are free of oscillatory artifacts. These results are nearly identical. For the FPS method, using the same time step, we see oscillatory artifacts when the reference velocity is 4500 m/s compared with the FPS result for $v_0 = 1500$ m/s. This shows that the FPS method is comparable to the second-order pseudo-analytical method but is more sensitive to the compensation velocity. The second-order pseudo-analytical method requires $\Delta t < \sqrt{6}\,\Delta x/\pi v$. Considering v = 4500 m/s, we find $\Delta t < 0.0026$ s, and the results for the second-order pseudo-analytical method are free of oscillatory noise independent of the compensation velocity as shown in Figs. 3(a) and 3(b). These results are much better then the ones obtained with the pseudo-analytical method [Figs. 2(d) and 2(e)] and they are free of numerical noise. Thus, these results show that the second order pseudo-analytical and the FPS (for the correct compensation velocity) and Lax-Wendroff methods are free

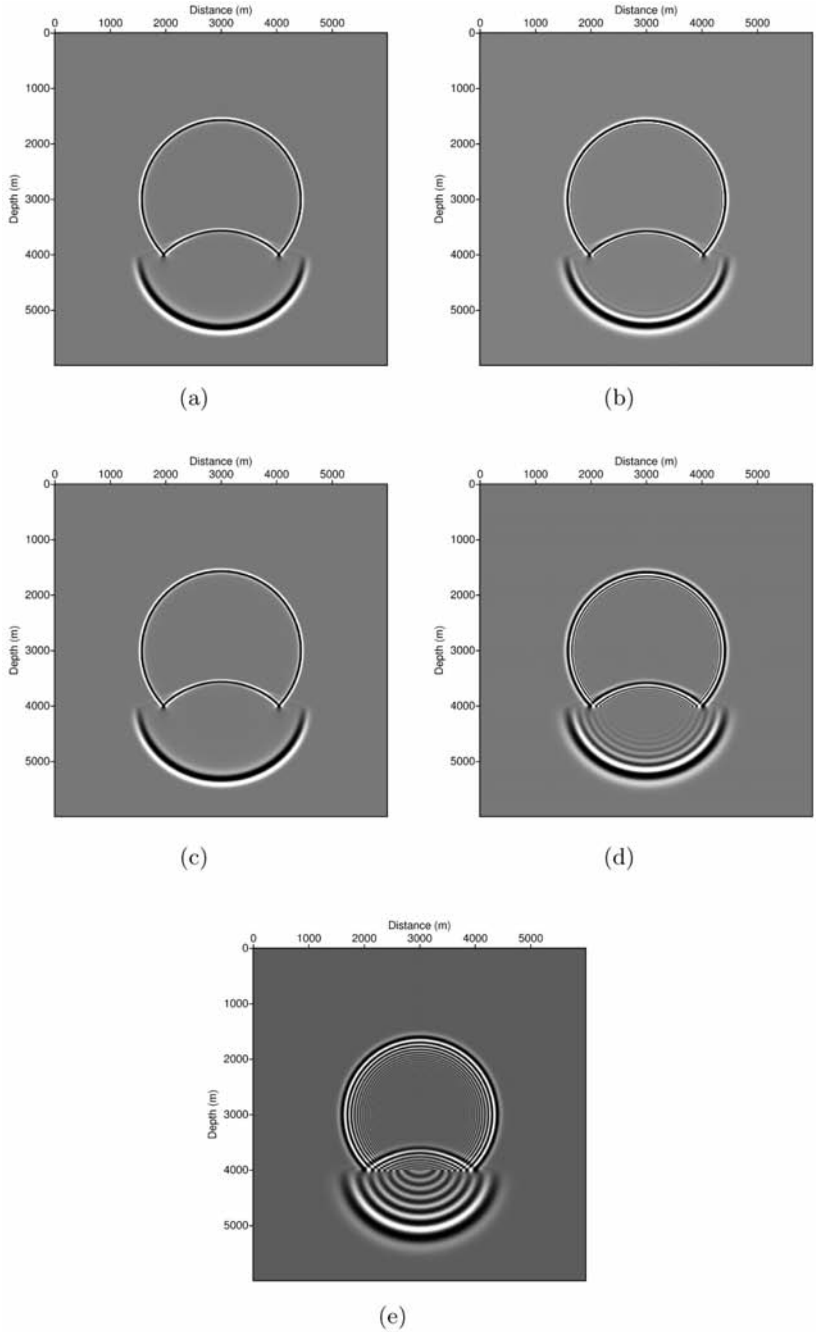

Fig. 2. Wavefield snapshot at time 1.0 s for a two-layer model: The upper layer velocity is 1500 m/s and bottom layer velocity is 4500 m/s. Pseudo-analytical method for: (a) $v_0 = 1500$ m/s and (b) $v_0 = 4500$ m/s with time step of $\Delta t = 0.001$ s. (c) $v_0 = 1500$ m/s and (d) $v_0 = 4500$ m/s, with $\Delta t = 0.0015$ s; and (e) $v_0 = 4500$ m/s, with $\Delta t = 0.0025$ s.

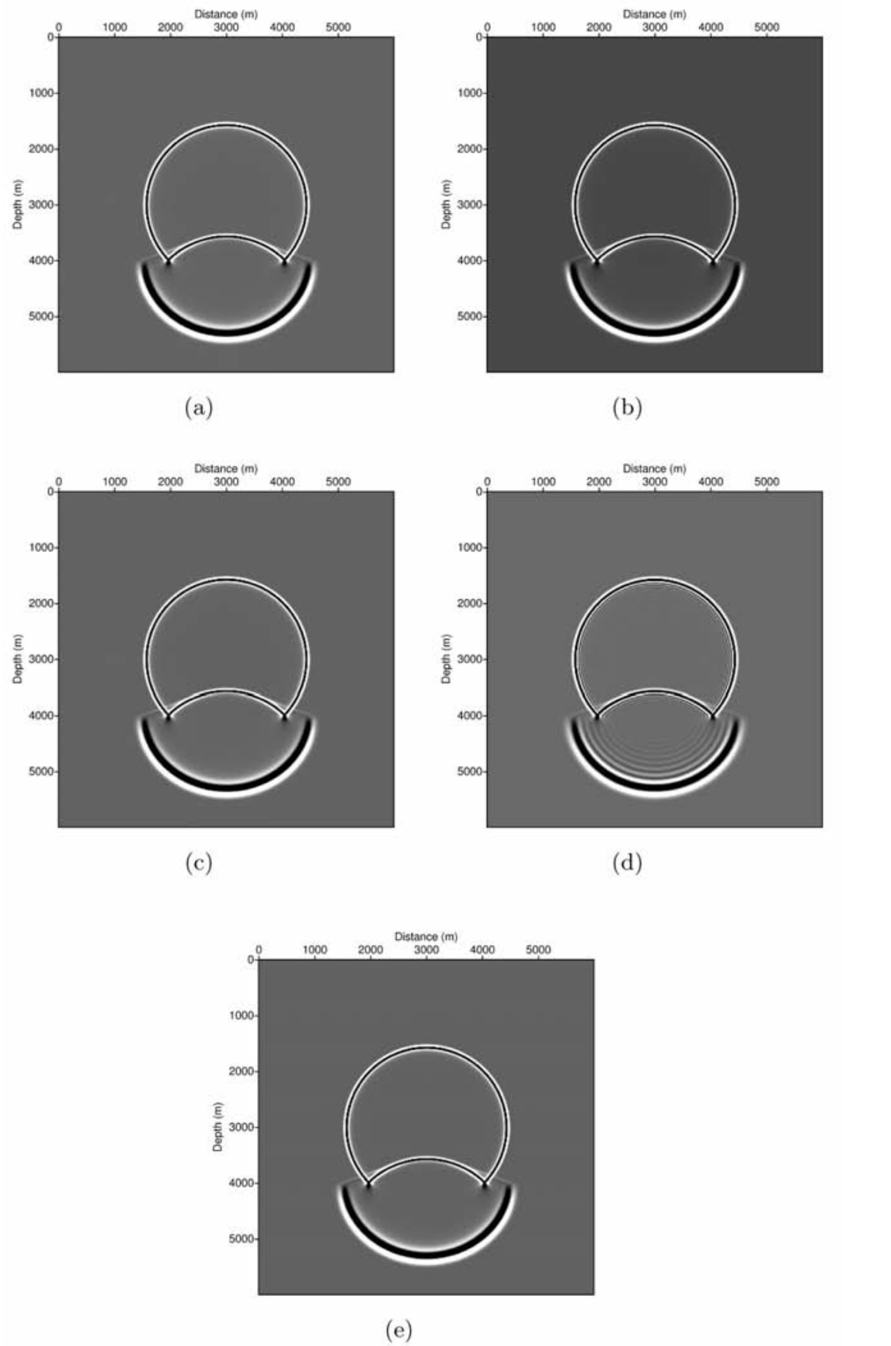

Fig. 3. Wavefield snapshot at time t = 1.0 s for two-layer model and time step of 0.0025 s, and taking $\alpha = 0.75$: Second order pseudo-analytical method with (a) $v_0 = 1500$ m/s and (b) $v_0 = 4500$ m/s; FPS method with (c) $v_0 = 1500$ m/s and (d) $v_0 = 4500$ m/s; (e) Fourth-order Lax-Wedroff method.

of these artifacts, even for larger time steps, and they provide efficient and accurate approximations for the Laplacian operator.

Fig. 4 shows results for the FPS method for a model with a vertical fault. The upper layer has a velocity of 1500 m/s and the lower layer's velocity is 4500 m/s. The velocity model is shown in the background of the Fig. 4(a). The source is located at (3000 m, 3000 m) and the grid spacing on this model is 15 m for both the horizontal and vertical directions. Taking the compensation velocity equal to the maximum velocity in the media, we have that $\alpha < 0.7$, which implies that $\Delta t < 0.0023$ s. As we can see in all these figures, for $\Delta t = 0.0025$ s and with the compensation velocities varying from 1500 to 4500 m/s, the numerical dispersion is still quite small.

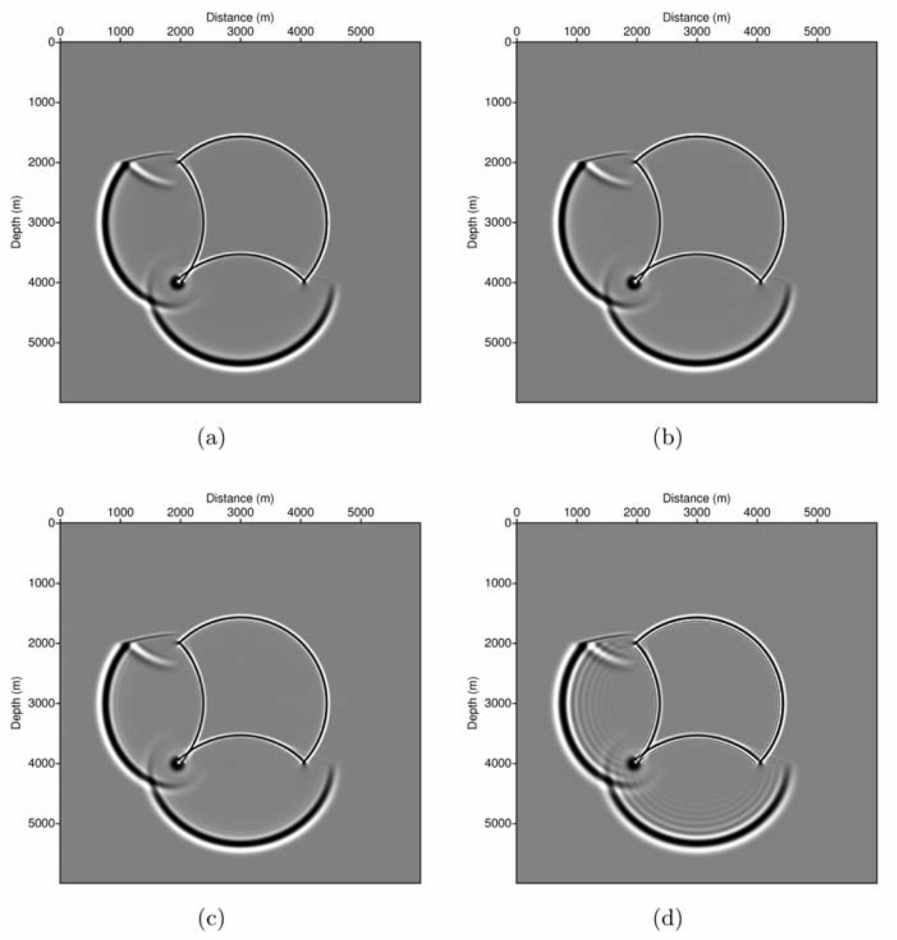

Fig. 4. Wavefield snapshot at time 1.0 s for a vertical fault mode for time step of $\Delta t = 0.0025$ s. FPS method with compensation velocity of: (a) $v_0 = 1500$ m/s; (b) $v_0 = 2500$ m/s; (c) $v_0 = 3500$ m/s; (d) $v_0 = 4500$ m/s. Taking the compensation velocity equal to maximum velocity in the model (4500 m/s), and $\Delta x = \Delta z = 15$ m, we obtain that $\Delta t < 0.0023$ s.

The result obtained with the compensation velocity equal to 3500 m/s [Fig. 4(c)], is free of oscillatory noise. [In this case, we note that for stability $\Delta t < 0.0034$ s, using $\Delta t < \sqrt{6}\Delta x/\pi\sqrt{(v^2 - v_0^2)}$]. However, for $v_0 = 4500$ m/s, the maximum time step is now very close the maximum time step allowed. Thus, the dispersion shows up again as we can see in Fig. 4(d).

Using now $\Delta t = 0.003$ s, the FPS method became unstable for $v_0 = 1500$ m/s and $v_0 = 2500$ m/s (Figs. 5(a) and 5(b), respectively). But for a compensation velocity of 3500 m/s [Fig. 5(c)] there is no dispersion and for $v_0 = 4500$ m/s [Fig. 5(d)], we again have oscillatory artifacts, because we have to have $\alpha < 0.7$ or $\Delta t < 0.0023$ to obtain stable results.

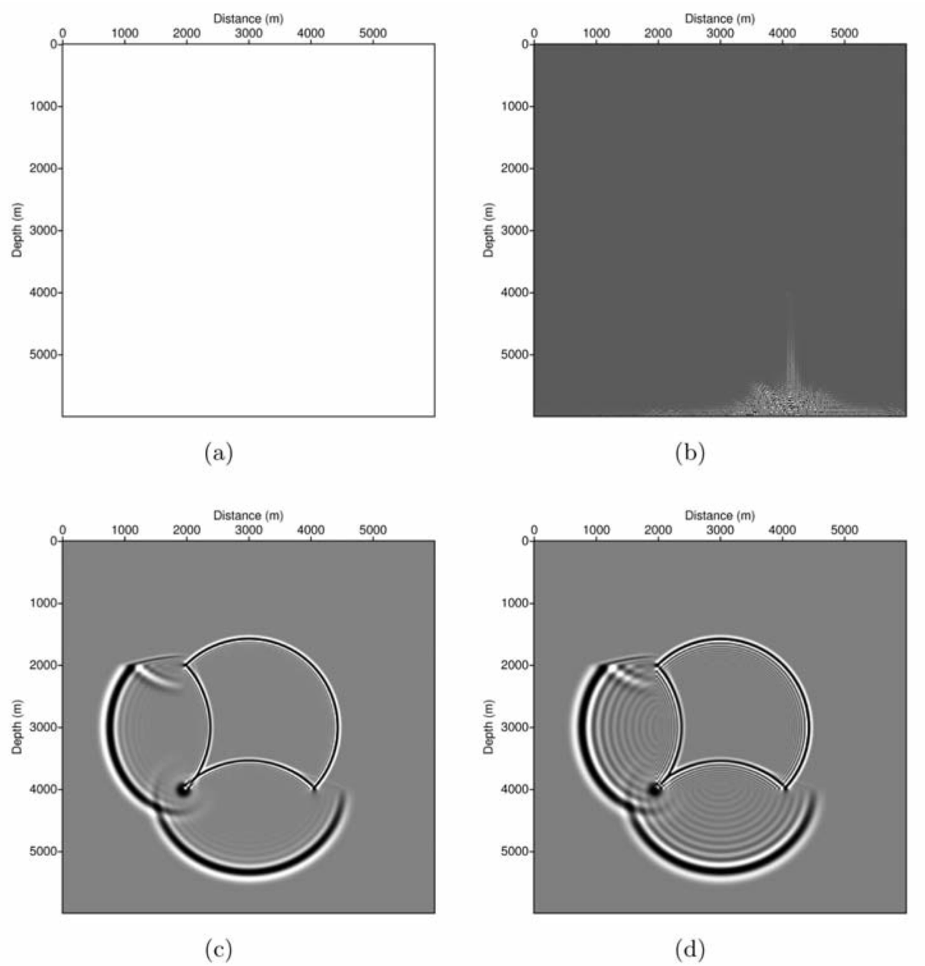

Fig. 5. Wavefield snapshot at time 1.0 s for a vertical fault mode for time step of $\Delta t = 0.003$ s. FPS method with compensation velocity of: (a) $v_0 = 1500$ m/s; (b) $v_0 = 2500$ m/s; (c) $v_0 = 3500$ m/s; (d) $v_0 = 4500$ m/s. Taking the maximum velocity in the model equal to 4500 m/s, and compensation velocity equal to $v_0 = 3500$ m/s, we obtain that $\Delta t < 0.0035$ s.

CONCLUSIONS

We have generalized the pseudo-analytical method to high-order formulations by introducing a new derivation. With this alternative derivation, the pseudo-analytical method can be interpreted as a modified Lax-Wendroff time stepping scheme: the latter method uses a truncated Taylor series while the former keeps the remainder term and approximates it in the wavenumber domain with constant velocities. The remainder term, by definition, is the pseudo-Laplacian operator. The sensitivity to the compensation velocity v_0 of the pseudo-Laplacians is inversely proportional to the order of accuracy. As a consequence, high-order pseudo-analytical methods are more accurate and therefore potentially more efficient than low-order ones for models with high velocity contrasts.

We have shown that both the second-order pseudo-analytical method and the Fourier finite difference method adopt the idea of pseudo-Laplacians to compensate for time stepping errors. Both methods contain one wavenumber domain term and one space domain term which leads to a two-step implementation. The wavenumber domain term of the Fourier finite difference method is a zero-th order pseudo-Laplacian. Consequently, it suffers from the same problem of being sensitive to the compensation velocity as the zero-th order pseudo-analytical method. In addition, the accuracy of the Fourier finite difference method is also dependent on the order of the finite difference operator employed. Thus, we used the pseudospectral method to evaluate the space domain term to improve the accuracy and therefore remove the effects of this finite difference approximation from our examples.

The computation cost of the second-order pseudo-analytical method, the Fourier finite difference method, and the fourth-order Lax-Wendroff time integration method is approximately same, when using the pseudospectral method to evaluate all spatial derivatives (the FFD method becomes the FPS method in this case). Both the second-order pseudo-analytical method, FFD and FPS method, however, have less restrictive stability conditions than the Lax-Wendroff fourth-order time stepping method. Compared with the second-order pseudo-analytical method, the stability and hence accuracy of the FPS is more dependent of the compensation velocity and care must be taken in its selection.

It is also possible to use other polynomials instead of a Taylor series to derive other variants of the pseudo-analytical method and Fourier finite-difference methods which may lead to higher accuracy and efficiency. But for the three methods studied here, we conclude that the second-order pseudo-analytical method is the best choice for high-accuracy numerical wave simulations in rapidly varying velocity media.

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