GENERALIZATION OF KIRCHHOFF REFLECTIVITY TO GO BEYOND MODELLING AND INVERSION OF FIRST-ORDER REFLECTION DATA - A THEORETICAL REVIEW.

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ABSTRACT

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I emphasize the connections and differences between Kirchhoff and Born modelling. I seize the opportunity to clarify aspects related to possibly non-smooth propagating media and the linearity approximation on reflectors. I discuss how they lead to a general expression for the conversion of a velocity perturbation into a reflectivity through the “generalized reflectivity" concept. The latter offers opportunities:

- On FWI approaches that include a reflectivity or least squares migration approaches that can be based on Kirchhoff or Born modelling: to rigorously convert the reflectivity into a velocity perturbation.
- In the framework of traditional Kirchhoff modelling scheme: to model first-order effects that go beyond first-order reflections (like first-order diffractions).
- In the framework of traditional Kirchhoff inversion or true amplitude migration, i.e., for the interpretation of seismic-migrated images: to give a basis to interpret by AVA (amplitude versus angle) more information than the amplitudes associated to first-order reflections, for instance the amplitudes of first-order diffractors. Also, it would theoretically allow to go beyond AVA analysis, inverting for the whole seismic image amplitude information (not only amplitude information at peaks) to recover the related velocity model perturbation. This is discussed formally in the article.

KEY WORDS: Reflectivity, migration, diffractions, Kirchhoff, Born, interpretation.

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INTRODUCTION

The aim of seismic imaging (Claerbout, 1985) is to characterize the geological structures of the subsurface from the analysis of seismic waves (Aki and Richards, 1980; Chapman, 2004). A central component of seismic imaging is the scale separation, i.e. the separation of a smooth background velocity containing the long wavelength components of the true subsurface velocity model from the short wavelength components (Claerbout, 1985; Tarantola, 2005). This separation is justified regarding the physical behavior of these two components with respect to band limited data (typically 3 to 80Hz) (Claerbout, 1985; Tarantola, 2005; Virieux and Lambaré, 2015): the background velocity exhibits a strongly non-linear behavior with respect to the data, affecting the kinematics of the seismic events, while the short wavelength components have a much more linear behavior, affecting mostly the amplitudes of the events. As a consequence, recovery of the background velocity and of the short wavelength components is usually done sequentially (Lailly, 1983; Tarantola, 1984). The first step is to compute the background velocity, typically by non-linear tomographic methods (Luo and Schuster, 1991). The second step is to compute the short wavelength components through a linear inversion process, considering first-order scattered events (reflections and diffractions) (Tarantola, 2005; Claerbout, 1971; Bleistein, 1987), called seismic migration or imaging. There are two ways in seismic migration of linearly representing the short wavelength components of the velocity model, usable in a least square version (Huang et al., 2016; Salomons et al., 2014):

- Using the Born approximation (Beylkin, 1985, 1986; Lambare et al., 1992; Bleistein et al., 2001), based on a velocity model perturbation.
- Using the Kirchhoff approximation (Claerbout, 1985; Bleistein et al., 2001; Stlok and De Hoop, 2002; ten Kroode et al., 1998; Brandsberg-Dahl et al., 2003), where the short wavelength components are represented through a reflectivity distribution, i.e., a volumetric distribution of reflection coefficients. Inspired by the pioneering work of (Beylkin, 1985, 1986). Bleistein's groundbreaking work (Bleistein, 1987; Bleistein et al., 2001) fundamentally establishes the reflectivity and shows how it can, at a later stage, be converted into material properties of the subsurface through an additional inversion process like AVA (amplitude versus angle) analysis (Bleistein, 1987; Russell, 1988).

Full waveform inversion (FWI) (Tarantola, 2005; Virieux and Operto, 2009) is another approach for characterizing the subsurface velocity. Its ultimate aim is to invert band-limited seismic data non-linearly for the full range of wavelength components of the velocity model. In common FWI applications, a local optimization scheme is used (each iteration being related to a linearization, i.e., the Born approximation) (Tarantola, 2005), so that an initial velocity model that is sufficiently good kinematically is needed to avoid local minima. Then, the non-linearity is sufficiently weak and FWI can invert for long wavelength components. A reflectivity can be introduced within FWI to model first-order reflections but it must be converted into a velocity perturbation at each iteration for the velocity update (Claerbout, 1985; Berkhout, 1982; Xu et al., 2012).

Those two representations of the short wavelength components of the velocity model, i.e., model perturbation (Born) versus reflectivity (Kirchhoff), are commonly used. They are both based on a linearization, but each offers some specifics. For instance, Born approximation allows for modelling of first-order reflections on weak discontinuities and first-order diffractions, whereas Kirchhoff approximation allows for modelling of first-order reflections possibly on stronger discontinuities and postcritical reflections. The connections and differences between the two have been studied from different points of view, see, e.g., (Bleistein et al., 2001; Ursin and Tygel, 1997; Alferini, 2002; Beydoun and Jin, 1994). In this paper, we propose to emphasize those connections and differences in a refreshing way.

First, we briefly recall the chain of approximations leading to Kirchhoff modelling equations. We seize the opportunity to clarify some aspects related to Kirchhoff modelling, concerning possibly non-smooth propagating media and the linearity approximation on reflectors. Then, we detail how Kirchhoff and Born modelling can lead to very similar expressions and how we can derive a general expression for the conversion from velocity model perturbation into reflectivity (and conversely) through a “generalized reflectivity" concept. We propose a different demonstration than the one existing in the literature (Ursin and Tygel, 1997). We then point out, from a formal point of view, the strengths and weaknesses of the Kirchhoff and Born modelling schemes.

The generalized reflectivity offers opportunities:

- In the framework of traditional Kirchhoff modelling scheme: to model first-order effects that go beyond first-order reflections (like first-order diffractions).
- On FWI approaches that include a reflectivity (Xu et al., 2012) or least squares migration approaches that can be based on Kirchhoff or Born modelling (Huang et al., 2016; Salomons et al., 2014): to rigorously convert the reflectivity into a velocity perturbation.
- In the framework of traditional Kirchhoff inversion or true amplitude migration, for the interpretation by AVA of more information than the amplitudes associated to first-order reflections, for instance the amplitudes of first-order diffractors. Also, it would theoretically allow us to go beyond AVA analysis, inverting for the whole seismic image amplitude information (not only amplitude information at peaks) to recover the related velocity model perturbation. This is discussed formally in this article.

KIRCHHOFF MODELLING AND INVERSION

In the following, t represents time, $\mathbf{r} = (x, y, z)$, position in the subsurface, $\mathbf{r}_s$, position of an impulsive source of signature $s(t)$ and $\mathbf{r}_r$, the receiver positions. Our time-direction Fourier transform convention is $A(\omega) = \int_{-\infty}^{+\infty} dt\, e^{-i\omega t} a(t)$. We use capital letters for the Fourier transform result.

Seismic waves are frequently modelled assuming a constant density acoustic approximation, i.e., using the scalar wave equation where the subsurface model is parameterized by the velocity. The subsurface wavefield $p(\mathbf{r}_s, \mathbf{r}, t)$ generated by a point source at $\mathbf{r}_s$ then obeys

for $\mathbf{r}_s \in \Re e^3, \forall \mathbf{r} \in \Re e^3$:

$$[\frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2} - \Delta]p(\mathbf{r}_s, \mathbf{r}, t) = \delta(\mathbf{r} - \mathbf{r}_s)s(t)$$

$$p(\mathbf{r}_s, \mathbf{r}, t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}p(\mathbf{r}_s, \mathbf{r}, t) = 0 \quad \text{for } t \leq 0. \tag{1}$$

c is the velocity of the subsurface. p is assumed to satisfy the Sommerfeld radiation condition.

Kirchhoff modelling

Reflectors (or “smooth physical interfaces") (Bleistein et al., 2001) are defined by discontinuities in the model c that generate reflections. Reflections are defined within 0-order geometrical optics (0-g.o.) or high-frequency approximation by the events that satisfy the Snell-Descartes law, which imposes a particular direction to a reflected ray according to the direction of the corresponding incident ray (Bleistein et al., 2001; Červený, 2001; Kravtsov and Orlov, 1990). (Contrariwise, diffraction events do not satisfy the Snell-Descartes law.)

In the following, we consider a subsurface composed of infinitely spread and sufficiently separated reflectors (in a sense that will be clarified later), with a sufficiently smooth velocity between reflectors from the 0-g.o. point of view. S_k denotes the position of one reflector surface in the subsurface. “Above S_k” means in the “incident" medium by slight abuse of language, see Fig. 1. The Green function $g(\mathbf{r}_s, \mathbf{r}, t)$ of the subsurface satisfies eq. (1) with $s(t) = \delta(t)$. It is decomposed above S_k into:

- An “incident" field g_{inc} that is generated by the source and does not interact with S_k and the medium below S_k; in other terms it satisfies eq. (1) above S_k with $s(t) = \delta(t)$ and radiation (or "absorbing" boundary) conditions on S_k.

- A field $g_{ref} = g - g_{inc}$ that represents what remains, i.e., reflections generated on S_k and events generated below S_k that "come back" into the medium above S_k. They are described through a boundary condition on S_k.

We briefly remind the main steps that lead to the Kirchhoff modelling approximation.

Firstly, the 0-g.o. approximation (Červený, 2001, Kravtsov and Orlov, 1990) for the Green function G_{inc} makes it possible to separate G_{inc} into the contributions related to each of the travel-time branches (or ray paths) that reach S_k for the source or receiver sides:

- The source-side Green function is

$$\forall \mathbf{r} \in S_k: G_{inc}(\mathbf{r}_s, \mathbf{r}, \omega) \approx \sum_{j \geq 1} G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega)$$
$$G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) = A^{(i)}(\mathbf{r}_s, \mathbf{r}) e^{-i\omega T^{(i)}(\mathbf{r}_s, \mathbf{r})}, \quad (2a)$$

where i denotes the travel-time branch numbers. $N(\mathbf{r}_s)$ denotes the number of direct travel-time branches (i.e., non-reflected, or refracted due to velocity inhomogeneities) and $i \leq N(\mathbf{r}_s)$ refers to these arrivals. $i > N(\mathbf{r}_s)$ refers to travel-time branches reflected (once or multiple times) within the medium above S_k but not on S_k (remembering that G_{inc} denotes the field that does not interact with S_k and the medium below). In eq. (2a), each wavefield on S_k related to a travel-time branch (i) is parameterized by an amplitude $A^{(i)}$ and travel-time $T^{(i)}$ that satisfy respectively the transport and eikonal equations (Červený, 2001, Kravtsov and Orlov, 1990). $\nabla_{\mathbf{r}} T^{(i)}(\mathbf{r}_s, \mathbf{r} \in S_k)$ defines the direction of the i^{th} source-side travel-time branch "ray" on S_k.

- The receiver-side Green function G_{ref} is defined in a similar way that in eq. (2a), with $\mathbf{r}_s \rightarrow \mathbf{r}_r$ and $i \rightarrow j$ in the notations above.

Fig. 1 gives an illustration, with $i > N(\mathbf{r}_s)$ for the source travel-time branch and $j \leq N(\mathbf{r}_r)$ for the receiver travel-time branch. An important quantity is the "incidence angle" $\theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})$ that is the acute angle at position $\mathbf{r} \in S_k$ between $\nabla_{\mathbf{r}} T^{(i)}$ and the unit vector $\mathbf{n}$ normal to S_k that points "downward" (well defined for smooth S_k only). $\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)$

denotes half the angle at reflector positions between the i^{th} source traveltime branch ray and the j^{th} receiver traveltime branch ray. For "specular" source and receiver ray pairs, i.e. reflections (that satisfy the Snell-Descartes law) (Bleistein et al., 2001; Ursin and Tygel, 1997), we have $\theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}) = \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)$.

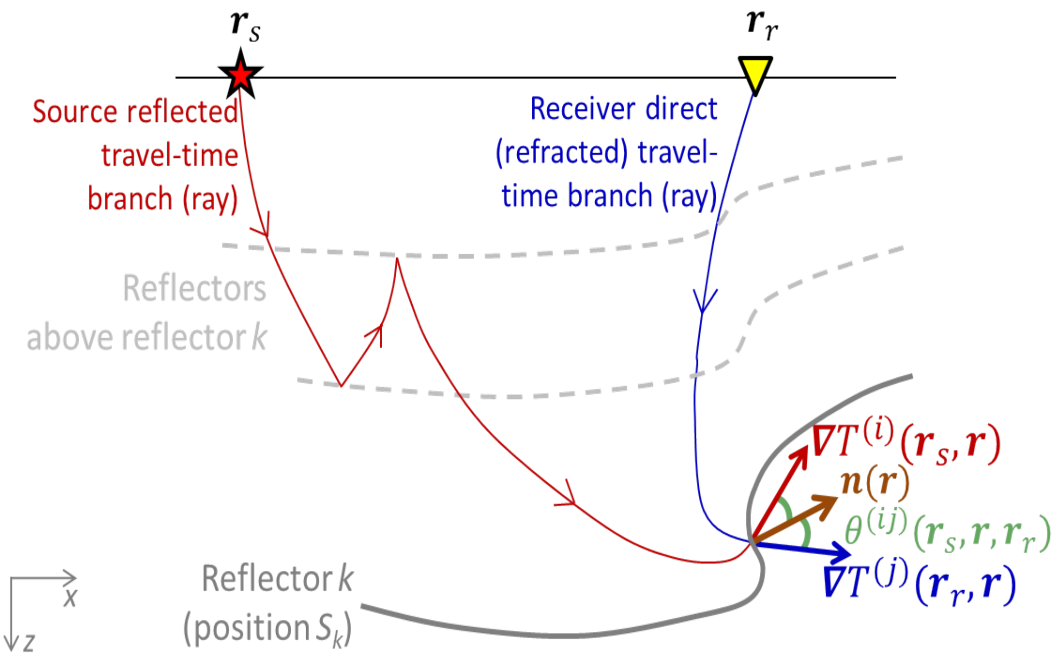

Fig. 1. Source and receiver travel-time branches related by a reflection "from above" on a reflector k.

Fig. 1 gives an illustration, with $i > N(\mathbf{r}_s)$ for the source travel-time branch and $j \leq N(\mathbf{r}_r)$ for the receiver travel-time branch. An important quantity is the "incidence angle" $\theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})$ that is the acute angle at position $\mathbf{r} \in S_k$ between $\nabla_{\mathbf{r}} T^{(i)}$ and the unit vector $\mathbf{n}$ normal to S_k that points downward (well defined for smooth S_k only). $\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)$ denotes half the angle at reflector positions between the i^{th} source travel-time branch ray and the j^{th} receiver travel-time branch ray. For specular source and receiver ray pairs, i.e. reflections (that satisfy the Snell-Descartes law) (Bleistein et al., 2001; Ursin and Tygel, 1997), we have $\theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}) = \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)$.

Secondly, the 0-g.o. (possibly complex) reflection coefficient on S_k is introduced (Červený, 2001; Kravtsov and Orlov, 1990; Bleistein et al., 2001; Berkhout, 1982)

$\forall \mathbf{r} \in S_k$:

$$R^{(i)}(\mathbf{r}_s, \mathbf{r}) = R\left(\mathbf{r}, \theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})\right) = \lim_{\epsilon \to 0^+} \frac{c_+(\mathbf{r})\cos(\theta_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r})) - \sqrt{c_-^2(\mathbf{r}) - \sin^2(\theta_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r}))c_+^2(\mathbf{r})}}{c_+(\mathbf{r})\cos(\theta_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r})) + \sqrt{c_-^2(\mathbf{r}) - \sin^2(\theta_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r}))c_+^2(\mathbf{r})}}$$

$$c_+(\mathbf{r}) = \lim_{\epsilon \to 0^+} c\big(\mathbf{r} + \epsilon \mathbf{n}(\mathbf{r})\big), \quad c_-(\mathbf{r}) = \lim_{\epsilon \to 0^+} c\big(\mathbf{r} - \epsilon \mathbf{n}(\mathbf{r})\big). \tag{2b}$$

Kirchhoff approximation considers only source and receiver travel-time branches coupled with a single reflection from above on S_k. This allows to relate linearly, within 0-g.o., G_{ref} to the reflection coefficients on S_k and to G_{inc}. $P_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega) = S(\omega)G_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega)$ denotes the total reflected wavefield measured at the earth's surface. $P_{ref,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega)$ represents the contribution of the i^{th} "source" travel-time branch, coupled with a single reflection from above on reflector k to the j^{th} "receiver" traveltime branch. In traditional Kirchhoff modelling approximation, we consider only the contributions related to direct (or refracted) source and receiver traveltime branches. Appendix A reminds the computation that leads to the Kirchhoff modelling approximation considering one reflector. Then, performing the linearity approximation on reflectors (supposing that the reflectors are in a configuration where they are separable almost everywhere, i.e. not too dense in a sense that will be clarified later), we sum each reflectors contribution and obtain the traditional Kirchhoff modelling approximation equation (Bleistein et al., 2001):

$$P_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega) \approx \sum_{k \geq 1} \sum_{i=1}^{N(\mathbf{r}_s)} \sum_{j=1}^{N(\mathbf{r}_r)} P_{ref,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega)$$

$$P_{ref,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega) = \int_{S_k} d\mathbf{r} \, ! \left(!, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r) \right) \frac{2\cos(\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r))}{c(\mathbf{r})} L_{inc}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \mathbf{r}, \omega)$$

$$L_{inc}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \mathbf{r}, \omega) = i\omega S(\omega) G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) G_{inc}^{(j)}(\mathbf{r}, \mathbf{r}_r, \omega). \tag{3}$$

All the performed approximations to obtain eq. (3) are valid for sufficiently high-frequencies (Bleistein et al., 2001), except the linearity approximation on reflectors that is physically valid for not too large velocity contrasts on the reflectors (even if eq. (3) is mathematically well defined for large contrasts). Subtleties about the linearity approximation on reflectors are discussed in Appendix A, related to non-direct travel-time branches ($i > N(\mathbf{r}_s)$ and $j > N(\mathbf{r}_r)$) and incident Green functions that can be different for different reflectors.

Interestingly, the stationary phase approximation allows the following replacement in eq. (3), even for non specular ray pair (Bleistein et al., 2001; Bleistein, 1987; Ursin and Tygel, 1997):

$$\forall \mathbf{r} \in S_k: \quad \theta^{(i)}_{inc}(\mathbf{r}_s, \mathbf{r}) \quad \Leftrightarrow \quad \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r). \tag{4}$$

The $G^{(i)}_{inc}$ were defined using 0-g.o., eq. (2), for the demonstration of the Kirchhoff modelling equation. But, as we factorized them in the final result, eq. (3), a wave propagation scheme can also be used for their computation. Ultimately, computing the $G^{(i)}_{inc}$ should involve the true subsurface velocity c, considering only a direct traveltime branch. This is not easy when c contains discontinuities. For instance within a 0-g.o. propagation this would imply resolving boundary conditions along each discontinuity in c. Within a wave propagation this would imply “muting" all reflections or the use of one-way propagators (Claerbout, 1985; Berkhout, 1982). To avoid the need for this, it is common practice to introduce in the modelling a smooth velocity $c_{inc} \leftarrow c$ that best reproduces travel-times and amplitudes of a wavefield generated at the earth’s surface and measured at the reflector positions. A smooth velocity c_{inc} that meets as well as possible those criteria can be defined through tomography (Woodward et al., 2008; Lambaré, 2008).

However, if a strong reflector (i.e., a large velocity contrast such as a salt dome) is present in the true subsurface, i.e., in c, the use of a unique smooth velocity c_{inc} will not be able to reproduce good amplitudes at positions below the reflector. A solution might involve considering two different smooth velocities: c^{above}_{inc} for propagations related to events occurring above the large contrast and c^{below}_{inc} for propagations related to events occurring below the large contrast. This would permit more freedom in the modelling to better reproduce the phase and amplitude below the reflector (Yarman et al., 2013). The considerations of Appendix A allow to fundamentally understand that this remains in the spirit of the most general form of the linearity approximation on reflectors, where the incident Green functions can be different for different reflectors.

Reflectivity distribution, Kirchhoff inversion and interpretation.

We convert the surface integral in eq. (3) into a volume integral to introduce the reflectivity $\hat{R}$, i.e., a volumetric distribution of the reflection coefficients. We use the "singular function of the reflector's surface", i.e. the Dirac delta distribution $\delta_{S_k}(\mathbf{r})$ that spikes on S_k, that satisfies $\int_{S_k} d\mathbf{r}\, A(\mathbf{r}) = \int_{\mathrm{V}} d\mathbf{r} A'(\mathbf{r})\delta_{S_k}(\mathbf{r})$, where A' is a well-behaved extension of A (defined only for $\mathbf{r} \in S_k$) in the whole volume V (that contains S_k). If $g_k(\mathbf{r}) = 0$ is an equation that defines the position of S_k, its singular function is defined by (Bleistein et al., 2001)

$$\delta_{S_k}(\mathbf{r}) = |\nabla g_k(\mathbf{r})|\delta\big(g_k(\mathbf{r})\big). \qquad (5)$$

We obtain (choosing V to be the space under the earth's surface located at $z = 0$) (Bleistein et al., 2001)

$$P_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega) \approx \sum_{i=1}^{N(\mathbf{r}_s)} \sum_{j=1}^{N(\mathbf{r}_r)} \int_{z\geq 0} d\mathbf{r}\, \hat{R}(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)) L_{inc}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \mathbf{r}, \omega)$$

$$\hat{R}\left(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)\right) = \textstyle\sum_{k\geq 1} R\left(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)\right) \frac{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)}{c_{inc}(\mathbf{r})} \delta_{S_k}(\mathbf{r}). \qquad (6)$$

Eq. (6) represents the Kirchhoff volumetric modelling equation (Bleistein et al., 2001). The reflectivity $\hat{R}$ represents a distribution that "points" on reflectors and contains information on the reflection coefficients R. The reflectivity concept becomes interesting in the context of Kirchhoff inversion. Suppose we recorded seismic data P at the earth's surface, pre-processed to retain only first-order reflections, and produced a smooth subsurface model c_{inc} that allows computing $L_{inc}^{(ij)}$. One can then invert the linear eq. (6) to recover the reflectivity $\hat{R}$. Let us consider only given travel-time branches, i.e., given i and j values (for instance the ones related to the shortest traveltimes). We return to $\theta_{inc}^{(i)}$ ($\Leftrightarrow \theta^{(ij)}$ for reflections) in eq. (6) for the practical purpose of removing the $\mathbf{r}_r$ dependency of the reflectivity and allow inversions per full shot, $\hat{R}(\mathbf{r}_s, \mathbf{r}) = \hat{R}(\mathbf{r}, \theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})) \Leftrightarrow \hat{R}(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r))$. One obtains the following linear inversion for each shot, i.e. $\forall \mathbf{r}_s$

$$\hat{R}_{inv}(\mathbf{r}_s, \mathbf{r}) = \\ \underset{\hat{R}(!_!,\mathbf{r})}{\arg\min} \int d\omega \int d\mathbf{r}_r |P(\mathbf{r}_s, \mathbf{r}_r, \omega) - \textstyle\int_{z\geq 0} d\mathbf{r} \hat{R}(\mathbf{r}_s, \mathbf{r}) L_{inc}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \mathbf{r}, \omega)|^2. \qquad (7)$$

This is called least-squares Kirchhoff inversion or true amplitude migration by shots (Claerbout, 1985; Tarantola, 2005; Bleistein, 1987). The reflectivity is rigorously speaking a singular distribution, but the result of the band-limited (limited ω range) and aperture-limited (limited $\mathbf{r}_r$ range) inversion (7) gives an estimate $\hat{R}_{inv}(\mathbf{r}_s, \mathbf{r})$ of the reflectivity "convolved" with a filter, that can be interpreted as a function. By making certain assumptions, amongst others one of real reflection coefficients, (Bleistein, 1987) shows that ($\hat{R}$ then has the dimension of a time divided by a squared distance)

$$\hat{R}_{inv}(\mathbf{r}_s, \mathbf{r}) \approx \sum_{k\geq 1} R(\mathbf{r}, \theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})) \left(\frac{2\cos(\theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}))}{c_{inc}(\mathbf{r})} \right)^2 \delta_{bl}(g_k(\mathbf{r}))$$

$$\delta_{bl}\big(g_k(\mathbf{r})\big) = \frac{1}{\pi} \Re e \int d\omega \, e^{i\omega g_k(\mathbf{r})} F(\omega), \tag{8}$$

where F represents the residual (band-limited) wavelet present in the data (after pre-processing). It maps in $\hat{R}_{inv}$ (also called reflectivity by slight abuse of language) through eq. (8). The band-limited Dirac δ_{bl} peaks where $g_k(\mathbf{r}) = 0$, i.e., on the reflector k positions. The inverted reflectivity $\hat{R}_{inv}$ is thus also called an image of the reflectors, or seismic image (Tarantola, 2005; Bleistein, 1987; Claerbout, 1985). F maps in the image in the direction perpendicular to the reflectors, see Appendix B. Of course enough frequency and receiver aperture ranges are needed so that the inversion (7) gives an unambiguous result, depending amongst others on the number of samples that describe the reflectivity. Eq. (8) allows us to deduce (Bleistein, 1987)

$$\forall \mathbf{r} \in S_k: \;\; \hat{R}_{inv}(\mathbf{r}_s, \mathbf{r}) \approx \alpha \, R\left(\mathbf{r}, \theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})\right) \left(\frac{2\cos(\theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}))}{c_{inc}(\mathbf{r})} \right)^2$$

$$\alpha = \frac{1}{\pi} \Re e \int d\omega \, F(\omega). \tag{9}$$

Suppose we could pick the amplitude variations along the amplitude peaks of continuous events in the image, i.e., along reflector positions S_k, and compute the incident angles θ_{inc} using rays or wavefield decomposition techniques and picked reflector dips. Then, using the definition of the reflection coefficient, eq. (2b), we can invert eq. (9) for c around reflector positions. This common method of interpretation of the seismic image is called "amplitude versus angle" (AVA) analysis (Russell, 1988; Bleistein, 1987).

Now we can clarify what we previously meant by reflectors in a not-too-dense configuration almost everywhere. From the Kirchhoff inversion point of view this means reflectors separated almost everywhere from each other by more than the source or receiver wavefield wavelengths at the dominant frequency.

GENERALIZATION OF THE KIRCHHOFF REFLECTIVITY

Born modelling approximation.

We now briefly recall the steps leading to Born modelling approximation (Bleistein et al., 2001). We consider eq. (1) and decompose the subsurface velocity c into

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2(\mathbf{r})} + \delta l(\mathbf{r}), \tag{10}$$

where c_0 is called a “reference" medium velocity, and δl is the squared slowness related to a “perturbation" of the reference medium. We decompose the subsurface wavefield into $P(\mathbf{r}_s, \mathbf{r}, \omega) = P_0(\mathbf{r}_s, \mathbf{r}, \omega) + P_{\delta l}(\mathbf{r}_s, \mathbf{r}, \omega)$, where the reference medium wavefield P_0 satisfies (in the time domain) a scalar wave equation like eq. (1) in medium c_0 with source wavelet $s(t)$, and $P_{\delta l}$ is the remaining wavefield (related to the perturbation δl). $P_{\delta l}$ can be decomposed into a linear contribution and a non-linear contribution (Bleistein et al., 2001), $P_{\delta l}(\mathbf{r}_s, \mathbf{r}_r, \omega) = P_L(\mathbf{r}_s, \mathbf{r}_r, \omega) + P_{NL}[P_{\delta l}](\mathbf{r}_s, \mathbf{r}_r, \omega)$, where (considering the earth’s surface at $z = 0$)

$$P_L(\mathbf{r}_s, \mathbf{r}_r, \omega) = -(i\omega)^2 S(\omega) \int_{z\geq 0} d\mathbf{r}\, \delta l(\mathbf{r}) G_0(\mathbf{r}_s, \mathbf{r}, \omega) G_0(\mathbf{r}, \mathbf{r}_r, \omega)$$
$$P_{NL}[P_{\delta l}](\mathbf{r}_s, \mathbf{r}_r, \omega) = -(i\omega)^2 \int_{z\geq 0} d\mathbf{r}\, \delta l(\mathbf{r}) P_{\delta l}(\mathbf{r}_s, \mathbf{r}, \omega) G_0(\mathbf{r}, \mathbf{r}_r, \omega). \tag{11}$$

G_0 denotes the causal Green function in the reference medium, i.e. satisfying eq. (1) with $c \to c_0$ and $s(t) = \delta(t)$. Eq. (11) does not involve any approximation. The Born approximation deals with the linear contribution P_L for the wavefield perturbation $P_{\delta l}(\mathbf{r}_s, \mathbf{r}, \omega) \approx P_L(\mathbf{r}_s, \mathbf{r}, \omega)$, which represents a good approximation if

$$\int_{z\geq 0} d\mathbf{r}\, \frac{1}{c_0^2(\mathbf{r})} \gg \left| \int_{z\geq 0} d\mathbf{r}\, \delta l(\mathbf{r}) \right| \text{ and } \frac{1}{c_0^2(\mathbf{r})} \gg |\delta l(\mathbf{r})|$$
$$\Rightarrow |P_L(\mathbf{r}_s, \mathbf{r}, \omega)| \gg |P_{NL}[P_{\delta l}](\mathbf{r}_s, \mathbf{r}, \omega)|. \tag{12}$$

This implies that c_0 remains close to c on average, i.e., reproduces travel-times of first-order scattered events.

Reformulation of Born modelling and generalized reflectivity.

We reformulate the Born modelling equation in a way that allows a direct comparison with the Kirchhoff modelling eq. (6). Demonstrations that share a similar spirit can be found in (Ursin and Tygel, 1997; Alferini, 2002). Here we propose a different demonstration that involves travel-time branches, and provide exhaustive detail on properties and consequences.

Firstly, as the Kirchhoff approximation involves 0-g.o. approximation, we introduce 0-g.o. for the propagation of G_0. Secondly, we must constrain the perturbation δl to describe at least all reflectors of the true subsurface model c; more generally, we constrain it to describe all discontinuities of the subsurface. As a consequence c_0 will be smooth and we can consider only direct (refracted) travel-time branches

$$G_0(\mathbf{r}_s, \mathbf{r}, \omega) \approx \sum_{j=1}^{N(\mathbf{r}_s)} G_0^{(j)}(\mathbf{r}_s, \mathbf{r}, \omega)$$
$$G_0^{(j)}(\mathbf{r}_s, \mathbf{r}, \omega) = A^{(j)}(\mathbf{r}_s, \mathbf{r}) e^{-i\omega T^{(j)}(\mathbf{r}_s, \mathbf{r})}. \quad (13)$$

We moreover constrain c_0 so that it reproduces the travel-times of the first-order events, or in other words, so that it minimizes the travel-time corrections present in P_{NL}. A velocity that meets this criterion as much as possible can be defined through tomography (Woodward et al., 2008; Lambaré, 2008). There is thus a close link between c_0 and the smooth velocity c_{inc} introduced in Kirchhoff modelling. So, we consider

$$c_0(\mathbf{r}) \Leftrightarrow c_{inc}(\mathbf{r}) \text{ and } G_0^{(j)}(\mathbf{r}_s, \mathbf{r}, \omega) \Leftrightarrow G_{inc}^{(j)}(\mathbf{r}_s, \mathbf{r}, \omega). \quad (14)$$

We next denote by $\mathbf{r}_{sr}(\mathbf{r}) = (x_{sr}(\mathbf{r}), y_{sr}(\mathbf{r}), z_{sr}(\mathbf{r}))$ any set of curvilinear coordinates obtained by transformation of the Cartesian coordinates $\mathbf{r} = (x, y, z)$; the superscript "sr" denotes that the curvilinear coordinates can be different for different $\mathbf{r}_s$ and $\mathbf{r}_r$ positions. The transformation must be well-defined, i.e. its Jacobian determinant must be non-null at every position $\mathbf{r}$. To that aim, the curvilinear abscissa must not cross. We choose a transformation $\mathbf{r}_{sr}^{(ij)}(\mathbf{r}) = (x_{sr}^{(ij)}(\mathbf{r}), y_{sr}^{(ij)}(\mathbf{r}), z_{sr}^{(ij)}(\mathbf{r}))$, where $z_{sr}^{(ij)}(\mathbf{r})$ is a curvilinear coordinate in the average direction of a direct ray that links $\mathbf{r}_s$ to $\mathbf{r}$, and of a direct ray that links $\mathbf{r}_r$ to $\mathbf{r}$, and $(x_{sr}^{(ij)}(\mathbf{r}), y_{sr}^{(ij)}(\mathbf{r}))$ are cartesian coordinates measured in the (x, y) system, see Fig. 2. The unit vector in the direction of the curvilinear abscissa $z_{sr}^{(ij)}(\mathbf{r})$ is (using standard rules of 0-g.o.)

$$\mathbf{e}_{sr}^{(ij)}(\mathbf{r}) = \frac{\nabla(T^{(i)}(\mathbf{r}_s,\mathbf{r}) + T^{(j)}(\mathbf{r}_r,\mathbf{r}))}{|\nabla(T^{(i)}(\mathbf{r}_s,\mathbf{r}) + T^{(j)}(\mathbf{r}_r,\mathbf{r}))|} \quad \Rightarrow \quad \frac{\partial}{\partial z_{sr}^{(ij)}(\mathbf{r})} = \mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla \qquad (15)$$

$$\frac{\partial}{\partial z_{sr}^{(ij)}(\mathbf{r})}(T^{(i)}(\mathbf{r}_!, \mathbf{r}) + T^{(j)}(\mathbf{r}_r, \mathbf{r})) = |\nabla(T^{(i)}(\mathbf{r}_s, \mathbf{r}) + T^{(j)}(\mathbf{r}_r, \mathbf{r}))| = \frac{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)}{c_{inc}(\mathbf{r})}.$$

The Jacobian matrix of the transformation is invertible because there is a unique incidence angle $\theta^{(ij)}$ at each position. Using eq. (13), we can show that for sufficiently large frequencies (i.e., for the high-frequency leading term) we have (the derivatives in the following are defined from the distributional derivative point of view)

$\forall \mathbf{r}$ such that $\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r) \neq \pi/2$:

$$G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) G_{inc}^{(j)}(\mathbf{r}, \mathbf{r}_r, \omega) \approx -\frac{1}{i\omega} \frac{\partial}{\partial z_{sr}^{(ij)}(\mathbf{r})} \left\{ \frac{c_{inc}(\mathbf{r})}{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)} G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) G_{inc}^{(j)}(\mathbf{r}, \mathbf{r}_r, \omega) \right\}. \qquad (16)$$

The area where $\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r) = \pi/2$ corresponds to the area where one ray can be directly traced between $\mathbf{r}_s$ and $\mathbf{r}_r$, i.e. to the “diving waves". To avoid related singularities, we constrain the Green functions in eq. (13) to contain no diving-wave travel-time branches. In areas where only diving waves occur in the subsurface, the Green functions $G_{inc}^{(i)}$ are thus null. This does not reduce the generality of our considerations as, using notations introduced above, the diving waves are described by the P_0 term while the first-order scattered events (first-order reflections and diffractions) are described by the P_L term. We also constrain without loss of generality δl to be null at the earth’s surface. We then insert eq. (16) in eq. (11) and use integration by parts

$$P_L(\mathbf{r}_s,\mathbf{r}_r,\omega) \quad (17)$$

$$\approx i\omega S(\omega)\sum_{i=1}^{N(\mathbf{r}_s)}\sum_{j=1}^{N(\mathbf{r}_r)}\int_{z\geq 0} d\mathbf{r}\,\delta l(\mathbf{r})\frac{\partial}{\partial z_{sr}^{(ij)}(\mathbf{r})}\{\frac{c_{inc}(\mathbf{r})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega)G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega)\}$$

$$\approx i\omega S(\omega)\sum_{i=1}^{N(\mathbf{r}_s)}\sum_{j=1}^{N(\mathbf{r}_r)}\int_{z\geq 0} d\mathbf{r}\,\{-\frac{\partial\delta l(\mathbf{r})}{\partial z_{sr}^{(ij)}(\mathbf{r})}\frac{c_{inc}(\mathbf{r})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}\}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega)G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega)$$

$$+i\omega S(\omega)\sum_{i=1}^{N(\mathbf{r}_s)}\sum_{j=1}^{N(\mathbf{r}_r)}\int_{z\geq 0} d\mathbf{r}\,\frac{\partial}{\partial z_{sr}^{(ij)}(\mathbf{r})}\{\delta l(\mathbf{r})\frac{c_{inc}(\mathbf{r})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega)G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega)\}$$

$$\approx i\omega S(\omega)\sum_{i=1}^{N(\mathbf{r}_s)}\sum_{j=1}^{N(\mathbf{r}_r)}\int_{z\geq 0} d\mathbf{r}\,\left\{-\frac{\partial\delta l(\mathbf{r})}{\partial z_{sr}^{(ij)}(\mathbf{r})}\frac{c_{inc}(\mathbf{r})}{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)}\right\}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega)G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega).$$

Details leading to the last relationship are given in this footnote[1]. This starts to look like the Kirchhoff modelling equation. Again, $\frac{\partial\delta l(\mathbf{r})}{\partial z_{sr}^{(ij)}(\mathbf{r})}$ is defined from the distributional derivative point of view. We rewrite eq. (17) as

$$P_L(\mathbf{r}_s,\mathbf{r}_r,\omega)\approx\sum_{i=1}^{N(\mathbf{r}_s)}\sum_{j=1}^{N(\mathbf{r}_r)}\int_{z\geq 0} d\mathbf{r}\,\hat{R}_{gen}\left(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)L_0^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\mathbf{r},\omega)$$

$$L_0^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\mathbf{r},\omega)=i\omega S(\omega)G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega)G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega)$$

$$\hat{R}_{gen}\left(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)=-\frac{c_{inc}(\mathbf{r})}{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)}\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla\delta l(\mathbf{r}). \quad (18)$$

This main result consists of a reformulation of the Born approximation using 0-g.o. (Ursin and Tygel, 1997; Alferini, 2002). It looks like the Kirchhoff modelling eq. (6), where $\hat{R}_{gen}$ is the counterpart of the Kirchhoff

---

[1] We denote by $J^{(ij)}$ the absolute value of the Jacobian determinant of the transformation $\mathbf{r}_{sr}^{(ij)}(\mathbf{r})=(x_{sr}^{(ij)}(\mathbf{r}),y_{sr}^{(ij)}(\mathbf{r}),z_{sr}^{(ij)}(\mathbf{r}))\rightarrow\mathbf{r}=(x,y,z)$. We have (with slight abuses of notation, using 0-g.o., keeping the high-frequency leading term in the penultimate line, using the Sommerfeld radiation condition for G_0 and that δl is null at the earth's surface in the last line)

$$\int_{z\geq 0} d\mathbf{r}\,\frac{\partial}{\partial z_{sr}^{(ij)}(\mathbf{r})}\{\delta l(\mathbf{r})\frac{c_{inc}(\mathbf{r})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega)G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega)\}$$

$$=\int_{z_{sr}^{(ij)}\geq 0} d\mathbf{r}_{sr}^{(ij)}\,J^{(ij)}\frac{\partial}{\partial z_{sr}^{(ij)}}\{\delta l(\mathbf{r}_{sr}^{(ij)})\frac{c_{inc}(\mathbf{r}_{sr}^{(ij)})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r}_{sr}^{(ij)},\mathbf{r}_r))}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r}_{sr}^{(ij)},\omega)G_{inc}^{(j)}(\mathbf{r}_{sr}^{(ij)},\mathbf{r}_r,\omega)\}$$

$$\approx\int_{z_{sr}^{(ij)}\geq 0} d\mathbf{r}_{sr}^{(ij)}\,\frac{\partial}{\partial z_{sr}^{(ij)}}\{J^{(ij)}\delta l(\mathbf{r}_{sr}^{(ij)})\frac{c_{inc}(\mathbf{r}_{sr}^{(ij)})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r}_{sr}^{(ij)},\mathbf{r}_r))}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r}_{sr}^{(ij)},\omega)G_{inc}^{(j)}(\mathbf{r}_{sr}^{(ij)},\mathbf{r}_r,\omega)\}$$

$$\approx\int dx_{sr}^{(ij)}dy_{sr}^{(ij)}\,[J^{(ij)}\delta l(\mathbf{r}_{sr}^{(ij)})\frac{c_{inc}(\mathbf{r}_{sr}^{(ij)})}{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r}_{sr}^{(ij)},\mathbf{r}_r))}G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r}_{sr}^{(ij)},\omega)G_{inc}^{(j)}(\mathbf{r}_{sr}^{(ij)},\mathbf{r}_r,\omega)]_{z_{sr}^{(ij)}=0}^{z_{sr}^{(ij)}=+\infty}$$

$$\approx 0.$$

reflectivity distribution. Note that the Kirchhoff approximation describes first-order reflections, whereas the Born approximation may contain more: it can describe any first-order events present in P_L (using notations introduced above) like first-order diffractions. We call $\hat{R}_{gen}$ “generalized reflectivity" even if it can describe more than reflections. We now have to understand precisely the differences between $\hat{R}_{gen}$ and Kirchhoff reflectivity $\hat{R}$

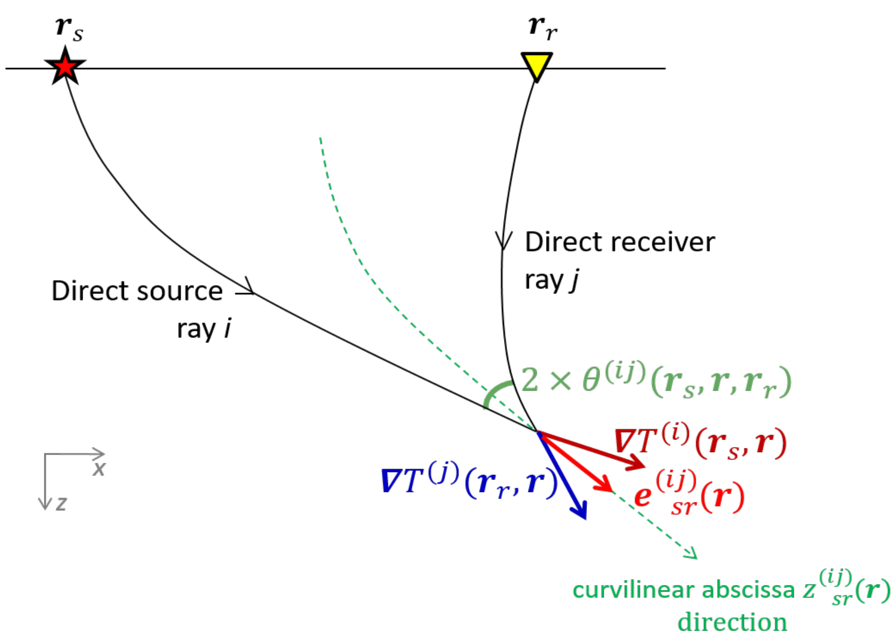

Fig. 2. $\theta^{(ij)}$, $\mathbf{e}_{sr}^{(ij)}$ and $z_{sr}^{(ij)}$ representation for a source and a receiver direct traveltime branches.

Link between Born generalized reflectivity and Kirchhoff reflectivity

We here firstly verify if Born generalized reflectivity $\hat{R}_{gen}$, eq. (18), reduces to the Kirchhoff reflectivity $\hat{R}$, eq. (6), when the perturbation contains only reflectors. We introduce the velocity perturbation δc defined by

$$c = c_{inc} + \delta c.$$

We have (using eqs. (10) and (14)) $\delta l = 1/(c_{inc} + \delta c)^2 - 1/c_{inc}^2$. Because δc must be sufficiently small, we can perform a 1st-order Taylor development and obtain $\delta l(\mathbf{r}) \approx -2\frac{\delta c(\mathbf{r})}{c_{inc}^3(\mathbf{r})}$. As c_{inc} is smooth and δc contains all the rapid velocity variations of the subsurface, we can consider

$$\nabla \delta l(\mathbf{r}) \approx -\frac{2}{c_{inc}^3(\mathbf{r})} \nabla \delta c(\mathbf{r}).$$

Inserting this result in eq. (18) we obtain $\hat{R}_{gen}$ as a function of the velocity perturbation for a smooth c_{inc}

$$\hat{R}_{gen}(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s\ \mathbf{r}!\mathbf{r}_r)) = \frac{1}{\cos(\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)) c_{inc}^2(\mathbf{r})} \mathbf{e}_{sr}^{(ij)}(\mathbf{r}). \nabla \delta c(\mathbf{r}). \quad (19)$$

Let us study what happens to the Born generalized reflectivity for a subsurface, i.e. a perturbation δc, composed only of reflectors. This can be modelled by

$$\delta c(\mathbf{r}) = \sum_{k \geq 1} a_k(\mathbf{r}) \left[H\left(g_k(\mathbf{r})\right) - 0.5 \right], \quad (20)$$

where H is the Heaviside function, and $a_k(\mathbf{r})$ a smooth (continuously differentiable) function with compact support that has the dimension of velocity, and simply “adjusts" the Heaviside jumps. When $\mathbf{r}$ is on reflector k, $a_k(\mathbf{r})$ equals the velocity jump $\Delta c(\mathbf{r})$ across the reflector (we use notation of eq. (2b) where $\mathbf{n}$ is the normal to the reflectors continuously extended between reflectors)

$$\begin{aligned} \forall \mathbf{r} \in S_k: \quad & a_k(\mathbf{r}) = \Delta c(\mathbf{r}) \\ & \Delta c(\mathbf{r}) = c_+(\mathbf{r}) - c_-(\mathbf{r}). \end{aligned} \quad (21)$$

Evaluating the reflection coefficient, eq. (2b), when Δc is small we obtain the linearized reflection coefficient R_{lin} (to 1st-order in δc)

$$R\left(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)\right) \xrightarrow{\text{small } \delta c} R_{lin}\left(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)\right) = \frac{\Delta c(\mathbf{r})}{2 c_{inc}(\mathbf{r}) \cos^2\left(\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)\right)} \quad (22)$$

We wish to compute the generalized reflectivity (19) corresponding to the velocity perturbation (20). We start by examining the $\mathbf{e}_{sr}^{(ij)}.\nabla \delta c$ term. We have

$$\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla\delta c(\mathbf{r}) = \sum_{k\geq 1} \mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla a_k(\mathbf{r})\big[H\big(g_k(\mathbf{r})\big) - 0.5\big]$$
$$= \textstyle\sum_{k\geq 1} a_k(\mathbf{r})\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla H(g_k(\mathbf{r}))$$
$$= \textstyle\sum_{k\geq 1} a_k(\mathbf{r})\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r})\frac{\partial}{\partial g_k}H(g_k(\mathbf{r}))$$
$$= \textstyle\sum_{k\geq 1} \Delta c(\mathbf{r})\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r})\delta\big(g_k(\mathbf{r})\big).$$

When inserted into eqs. (18) and (19), the contribution of the $\sum_{k\geq 1} \mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla a_k(\mathbf{r})[H(g_k(\mathbf{r})) - 0.5]$ term is negligible compared to the contribution of the $\sum_{k\geq 1} \Delta c(\mathbf{r})\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r})\delta(g_k(\mathbf{r}))$ term. Indeed, the second term leads to a sum of surface integrals [because of $\delta(g_k(\mathbf{r}))$] for which stationary phases exist for reflections whereas the first term (where a_k is smooth) leads to a sum of volume integrals for which stationary phases exist for diving waves only. As the Green functions $G_{inc}^{(i)}$ entering into our Born modelling equation were constrained to contain no diving wave, the impact of the second term can be neglected in our case. The dominant contribution of $\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla\delta c(\mathbf{r})$ is thus given by [using eq. (22)]

$$\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla\delta c(\mathbf{r})$$
$$\rightarrow \sum_{k\geq 1} \Delta c(\mathbf{r})\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r})\delta(g_k(\mathbf{r}))$$
$$= \sum_{k\geq 1} R_{lin}(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))2c_{inc}(\mathbf{r})\cos^2(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r})\delta(g_k(\mathbf{r})).$$

Inserting this result in eq. (19) gives

$$\hat{R}_{gen}\left(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right) =$$
$$\textstyle\sum_{k\geq 1} R_{lin}(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))\frac{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}{c_{inc}(\mathbf{r})}\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r})\delta(g_k(\mathbf{r})). \quad (23)$$

We can consider, again from stationary phase reasoning (Bleistein et al., 2001; Bleistein, 1987; Ursin and Tygel, 1997), $\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla g_k(\mathbf{r}) \Leftrightarrow |\nabla g_k(\mathbf{r})|$. Finally, using eq. (5), we obtain

$$\hat{R}_{gen}(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)) = \textstyle\sum_{k\geq 1} R_{lin}(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))\frac{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}{c_{inc}(\mathbf{r})}\delta_{S_k}(\mathbf{r}). \quad (24)$$

Gathering previous results we have

<u>Born generalized reflectivity:</u>

$$\hat{R}_{gen}(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)) \tag{25}$$

$$= \frac{1}{\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))c_{inc}(\mathbf{r})}\mathbf{e}_{sr}^{(ij)}(\mathbf{r}).\nabla\delta c(\mathbf{r})$$

Describes first-order effects related to any kind of small perturbations δc (first-order reflections and diffractions).

$$\xrightarrow{\text{reflectors only}} \sum_{k\geq 1} R_{lin}\left(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)\frac{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)}{c_{inc}(\mathbf{r})}\delta_{S_k}(\mathbf{r})$$

$$\text{with}\quad \theta^{(\mathrm{ij})}(\mathbf{r}_\mathrm{s},\mathbf{r},\mathbf{r}_\mathrm{r}) \Leftrightarrow \theta_{\mathrm{inc}}^{(\mathrm{i})}(\mathbf{r}_\mathrm{s},\mathbf{r})$$

Describes first-order reflections related to sufficiently weak discontinuities.

For comparison let us remember the content of Kirchhoff reflectivity, eq. (6)

<u>Kirchhoff !"#$"%&'('&)!</u>

$$\hat{R}(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)) \tag{26}$$

$$= \sum_{k\geq 1} R\left(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)\frac{2\cos\left(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)\right)}{c_{inc}(\mathbf{r})}\delta_{S_k}(\mathbf{r})$$

$$\text{with}\quad \theta^{(\mathrm{ij})}(\mathbf{r}_\mathrm{s},\mathbf{r},\mathbf{r}_\mathrm{r}) \Leftrightarrow \theta_{\mathrm{inc}}^{\mathrm{i}}(\mathbf{r}_\mathrm{s},\mathbf{r})$$

Describes first-order (possibly postcritical) reflections related to (possibly larger) velocity discontinuities.

Let us discuss what we have learned. From the propagation point of view, Born c_0 has been constrained here to be very close to Kirchhoff's c_{inc}. According to previous considerations and Appendix A, Kirchhoff has a slight advantage over Born because Kirchhoff modelling may be more “effective" (the general form of the linearity approximation on reflectors allows different modellings for events above and below strong reflectors). From the reflectivity point of view, we notice main differences if we compare Born (25) to Kirchhoff (26):

1. For reflections: Born is based on a linearized version R_{lin} of the 0-g.o. reflection coefficient R, whereas Kirchhoff is based on the full 0-g.o. reflection coefficient R defined by eq. (2b). This represents an advantage for Kirchhoff in describing the following:

(a) reflections due to larger velocity contrasts,
(b) larger incidence angle reflections,
(c) complex reflections.

These effects affect the phase of the wavefield, and are thus contained in the non-linear term P_{NL} (using notations introduced above). Because the Born approximation does not account for it, these effects cannot be included in Born generalized reflectivity. As those effects that are non-linear with respect to the wavefield perturbation are still linear with respect to the full 0-g.o. reflection coefficient, they may be accounted for by Kirchhoff. But remind that the linearity approximation on reflectors physically implies not too large velocity contrasts, limiting the ranges of validity of points (a)-(c).

2. For very dense reflector configurations: The generalized reflectivity remains well defined in the limit of a configuration where it is no longer possible to separate each reflector almost everywhere.
3. For other first-order events (like first-order diffractions): Born can describe them whereas traditional Kirchhoff cannot. The price to pay is that the reflectivity must then depend explicitly on the receiver positions $\mathbf{r}_r$ through $\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)$ (obvious for diffractions because they radiate in every direction).

According to the first point, Kirchhoff contains more than Born. According to the second and third points, Born contains more than Kirchhoff. The considerations of this article allow us to gather the strengths of both schemes in a unique scheme: one may use Born's generalized reflectivity $\hat{R}_{gen}$ together with the full reflection coefficient R instead of the linearized reflection coefficient R_{lin} to model the reflections (only).

SOME THEORETICAL CONSIDERATIONS ON THE UTILITY OF THE GENERALIZED REFLECTIVITY FOR INTERPRETATION.

The Kirchhoff inversion procedure has more flexibility than the direct inversion of the Born modelling equation. Indeed, the latter attempts to directly recover a material property of the subsurface δl (or δc), which is independent of the source position. The least-squares inversion of Born modelling eq. (11) is a procedure that combines the data from all sources (P represents data recorded at the earth's surface, pre-processed to retain only first-order events)

$$\min_{\delta l(\mathbf{r})} \int d\mathbf{r}_s \int d\omega \int d\mathbf{r}_r \,|P(\mathbf{r}_s, \mathbf{r}_r, \omega) + (i\omega)^2 S(\omega) \int_{z\geq 0} d\mathbf{r}\, \delta l(\mathbf{r}) G_0(\mathbf{r}_s, \mathbf{r}, \omega) G_0(\mathbf{r}, \mathbf{r}_r, \omega)|^2. \qquad (27)$$

Kirchhoff inversion attempts to recover a reflectivity (or a seismic image) that is not a material property of the subsurface and depends on the source position. Thus Kirchhoff least-squares inversion, eq. (7), is done for each shot (or offset if the data are reorganized) independently. As a second step, the material properties of the subsurface are recovered by inverting the obtained reflectivity. This step has already been presented above to interpret amplitudes of the reflectors present in the seismic image. In the following we discuss how the generalized reflectivity concept allows us to interpret events other than reflectors in the seismic image.

Suppose we recorded data P at the earth's surface and pre-processed to retain only first-order events, with non-linear (or higher-order) events filtered out (in particular multiple reflections). Suppose we also produced a smooth subsurface model c_{inc} that allows us to compute $G_{inc}^{(i)}$ and $\theta^{(ij)}$ for chosen direct travel-time branches. Traditional Kirchhoff inversion, eq. (7), inverts for a reflectivity $\hat{R}_{inv}(\mathbf{r}_s, \mathbf{r})$ that does not depend on the receiver positions $\mathbf{r}_r$ (Bleistein, 1987). As the generalized reflectivity fits into a Kirchhoff modelling framework, eq. (25) can directly be used to interpret the result of a Kirchhoff inversion (7). Or course the most exhaustive scheme to invert for the generalized reflectivity should depend on the receiver positions because $\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)$ is included in the definition of the generalized reflectivity. This would not cause any formal difficulties but would lead to an inversion scheme that is different from the commonly implemented one, eq. (7), with additional ill-posed problems. Here we want our considerations to be applicable to the commonly implemented scheme. The obtained $\hat{R}_{inv}$ then represents a band-limited average of $\hat{R}_{gen}$, eq. (25), over the receiver positions. If S_{rec} denotes the area over which the receiver are spread, we have

$$\hat{R}_{inv}(\mathbf{r}_s, \mathbf{r}) \approx \frac{1}{S_{rec}} \int_{S_{rec}} d\mathbf{r}_r \, \hat{R}_{gen}^{bl}(\mathbf{r}, \theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)) \approx \mathbf{a}(\mathbf{r}_s, \mathbf{r}).\nabla \delta c_{bl}(\mathbf{r})$$

$$\mathbf{a}(\mathbf{r}_s, \mathbf{r}) = \frac{1}{c_{inc}^2(\mathbf{r})} \frac{1}{S_{rec}} \int_{S_{rec}} d\mathbf{r}_r \, \frac{1}{\cos(\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r))} \mathbf{e}_{sr}^{(ij)}(\mathbf{r}), \qquad (28)$$

where δc_{bl} is the band-limited version of the velocity perturbation (because in practice we deal with band-limited Kirchhoff inversion). Inverting eq. (28) for each shot would allow us to convert the seismic image (i.e. the reflectivity $\hat{R}_{inv}$ computed by traditional Kirchhoff inversion schemes) into a velocity perturbation δc_{bl} and vice-versa. More generally one could also invert eq. (28) for each source-receiver configuration. This offers opportunities for further interpretation of seismic images and also for FWI approaches that include a reflectivity, showing how to rigorously convert the reflectivity into a velocity perturbation.

We discussed above the traditional way to interpret seismic images, considering only the amplitudes of reflectors. But seismic data and images also contain events that are not related to reflections, for instance those related to diffractions. It has been kinematically understood why migration collapses first-order diffractions in the seismic image (Claerbout, 1985; Aki and Richards, 1980), and the generalized reflectivity considerations explain this from a fundamental point of view. Moreover, eq. (28) can be used to interpret events related to first-order diffractions, i.e., recover the perturbation δc_{bl} that generates them. If we deal with a stack over shots, we have from eq. (28)

$$\int d\mathbf{r}_s\, \hat{R}_{inv}(\mathbf{r}_s, \mathbf{r}) \approx \alpha(\mathbf{r})\, \mathbf{e}_{average}(\mathbf{r}). \nabla \delta c_{bl}(\mathbf{r})$$

$$\mathbf{e}_{average}(\mathbf{r}) = \frac{1}{\alpha(\mathbf{r})} \mathbf{a}_{stack}(\mathbf{r}) \text{ with } \alpha(\mathbf{r}) = \sqrt{\mathbf{a}_{stack}(\mathbf{r}). \mathbf{a}_{stack}(\mathbf{r})}$$

$$\mathbf{a}_{stack}(\mathbf{r}) = \frac{1}{c_{inc}^2(\mathbf{r})} \frac{1}{S_{rec}} \int_{S_{rec}} d\mathbf{r}_r \int d\mathbf{r}_s \frac{1}{\cos(\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r))} \mathbf{e}_{sr}^{(ij)}(\mathbf{r}). \qquad (29)$$

The direction of the unit vector $\mathbf{e}_{average}$ depends on the acquisition configuration and the propagation in the subsurface (through the "illumination direction" $\mathbf{e}_{sr}^{(ij)}$ and $\theta^{(ij)}$). As previously mentioned, for reflectors, a residual wavelet maps in the direction perpendicular to them whatever the acquisition. For spike diffractors, the wavelet will map in the $\mathbf{e}_{average}$ direction in the image stacked over shots, this direction depending on the acquisition configuration. We denote by $\eta(\mathbf{r})$ the curvilinear abscissa defined by $\mathbf{e}_{average}(\mathbf{r})$ and consider the transformation $\mathbf{r} = (x, y, z) \rightarrow (x_s(\mathbf{r}), y_s(\mathbf{r}), \eta(\mathbf{r}))$, where $x_s(\mathbf{r})$ and $y_s(\mathbf{r})$ define the positions at the earth surface associated to $\eta(\mathbf{r})$. We have

$$\mathbf{e}_{average}(\mathbf{r}).\nabla = \frac{\partial}{\partial\eta(\mathbf{r})}. \tag{30}$$

From eqs. (29) and (30), one has to integrate $\frac{1}{\alpha(\mathbf{r})}\int d\mathbf{r}_s\, \hat{R}_{inv}(\mathbf{r}_s, \mathbf{r})$ over the curvilinear abscissa η to recover the band-limited velocity perturbation δc_{bl}. We obtain, with slight abuse of notation

$$\boxed{\delta c_{bl}(\mathbf{r}) = \int_0^{\eta(\mathbf{r})} d\eta' \, \frac{1}{\alpha(x_s(\mathbf{r}), y_s(\mathbf{r}), \eta')}\int d\mathbf{r}_s\, \hat{R}_{inv}(\mathbf{r}_s; x_s(\mathbf{r}), y_s(\mathbf{r}), \eta') + constant.} \tag{31}$$

$\mathbf{e}_{average}$, eq. (29), is computable through the knowledge of the propagation directions of the source and receiver wavefields. The $constant$ can be computed by imposing that $\delta c_{bl}(\mathbf{r})$ is null at the earth's surface. Thus the interpretation of more information than that associated with reflectors is theoretically possible using the generalized reflectivity.

CONCLUSIONS

We recalled the chain of approximations leading to Kirchhoff and Born modelling equations. They both contain a linearization but each offers some specifics. The Kirchhoff approximation allows, for example, the modelling of first-order reflections on stronger discontinuities and postcritical reflections. Born approximation makes it possible to model reflections on weak discontinuities only, but also first-order events beyond reflections (like first-order diffractions). We pointed out from a fundamental point of view the strengths and weaknesses of these schemes.

We took the opportunity to clarify some aspects related to Kirchhoff modelling approximation, concerning possibly non-smooth propagating media and the linearity approximation on reflectors. We discussed how Kirchhoff and Born modelling lead a general expression for the conversion from velocity model perturbation to reflectivity (and conversely) through the generalized reflectivity concept.

The generalized reflectivity offers opportunities that have been discussed formally in the article:

- On FWI approaches that include a reflectivity or least squares migration approaches that can be based on Kirchhoff or Born modelling: to rigorously convert the reflectivity into a velocity perturbation.

- In the framework of traditional Kirchhoff inversion or true amplitude migration: to interpret by AVA (amplitude versus angle) more information than the amplitudes associated to first-order reflections in seismic-migrated images, for instance the amplitudes of first-order diffractors. Also, it would theoretically allow us to go beyond AVA analysis, inverting for the whole seismic image amplitude information (not only amplitude information at peaks) to recover the related velocity model perturbation.
- In the framework of traditional Kirchhoff modelling scheme: to model first-order effects that go beyond first-order reflections (like first-order diffractions).

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APPENDIX A

KIRCHHOFF MODELLING APPROXIMATION AND LINEARITY APPROXIMATION ON REFLECTORS

Considering previously introduced notations and the representation theorem (Aki and Richards, 1980; Bleistein et al., 2001), some manipulations allow us to demonstrate the so-called "Kirchhoff integral" (that does not involve any approximation)

$$G_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega) = \int_{S_k} d\mathbf{r} \left(\nabla G_{ref}(\mathbf{r}_s, \mathbf{r}, \omega) G_{inc}(\mathbf{r}, \mathbf{r}_r, \omega) - G_{ref}(\mathbf{r}_s, \mathbf{r}, \omega) \nabla G_{inc}(\mathbf{r}, \mathbf{r}_r, \omega) \right) . \mathbf{n}(\mathbf{r}). \quad \text{(A-1)}$$

The Kirchhoff approximation allows us to ease the resolution of eq. (A-1) by finding approximate G_{ref} and $\mathbf{n}. \nabla G_{ref}$ values for those that enter into the integral. The essence is to assume the following relationship between the Green functions $G_{inc}^{(i)}$ and $G_{!"\#}^{!i)}$ along a reflector S_k (Bleistein et al.,2001)

$$\forall \mathbf{r} \in S_k \text{: } G_{ref}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) \approx R^{(i)}(\mathbf{r}_s, \mathbf{r}) G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega)$$

$$\nabla G_{ref}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega). \mathbf{n}(\mathbf{r}) \approx -R^{(i)}(\mathbf{r}_s, \mathbf{r}) \nabla G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega). \mathbf{n}(\mathbf{r}), \quad \text{(A-2)}$$

where $R^{(i)}$ the reflection coefficient defined in eq. (2b). Inserting eq. (A-2) in eq. (A-1), using eq. (2a) and keeping the high-frequency leading terms gives (Bleistein et al., 2001) $G_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega) = \sum_{i,j \geq 0} G_{ref}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega)$, where

$$G_{ref}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega) = i\omega \int_{S_k} d\mathbf{r} \, R\left(\mathbf{r}, \theta_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r})\right) \mathbf{n}(\mathbf{r}). \nabla(T^{(i)}(\mathbf{r}_s, \mathbf{r}) + T^{(j)}(\mathbf{r}_r, \mathbf{r}))$$

$$\times G_{inc}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) G_{inc}^{(j)}(\mathbf{r}, \mathbf{r}_r, \omega). \quad \text{(A-3)}$$

Using the property that the phase of this equation is stationary when the Snell-Descartes law for reflections is satisfied (Bleistein et al., 2001; Ursin and Tygel, 1997), we can make the following replacements in eq. (A-3) for sufficiently high frequencies (Bleistein et al., 2001; Bleistein, 1987; Ursin and Tygel, 1997):

$$\forall \mathbf{r} \in S_k: \quad \theta_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r}) \Leftrightarrow \theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)$$

$$R(\mathbf{r},\theta_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r})) \Leftrightarrow R(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))$$

$$\mathbf{n}(\mathbf{r}).\nabla\left(T^{(i)}(\mathbf{r}_s,\mathbf{r})+T^{(j)}(\mathbf{r}_r,\mathbf{r})\right)$$

$$\Leftrightarrow |\nabla(T^{(i)}(\mathbf{r}_s,\mathbf{r})+T^{(j)}(\mathbf{r}_r,\mathbf{r}))| = 2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))/c(\mathbf{r}).$$

By inserting those results in eq. (A-3) and factorizing the 0-g.o.-Green functions, we obtain

$$P_{ref}(\mathbf{r}_s,\mathbf{r}_r,\omega) = \sum_{i,j\geq 0} P_{ref}^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\omega)$$

$$P_{ref}^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\omega) = \int_{S_k} d\mathbf{r} R(\mathbf{r},\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r)) \frac{2\cos(\theta^{(ij)}(\mathbf{r}_s,\mathbf{r},\mathbf{r}_r))}{c(\mathbf{r})} L_{inc}^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\mathbf{r},\omega)$$

$$L_{inc}^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\mathbf{r},\omega) = i\omega S(\omega) G_{inc}^{(i)}(\mathbf{r}_s,\mathbf{r},\omega) G_{inc}^{(j)}(\mathbf{r},\mathbf{r}_r,\omega), \quad \text{(A-4)}$$

where $P_{ref}(\mathbf{r}_s,\mathbf{r}_r,\omega) = S(\omega)G_{ref}(\mathbf{r}_s,\mathbf{r}_r,\omega)$ denotes the total wavefield reflected on S_k and measured at the earth's surface. Each $P_{ref}^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\omega) = S(\omega)G_{ref}^{(ij)}(\mathbf{r}_s,\mathbf{r}_r,\omega)$ is related to a single reflection event on S_k for the corresponding source and receiver travel-time branches. This is the so-called Kirchhoff modelling approximation equation for one reflector S_k. Several extensions exist (Červený, 2001; Kravtsov and Orlov, 1990; Brandsberg-Dahl et al., 2003; ten Kroode et al., 1998; Stlok and De Hoop, 2002; Beylkin, 1985, 1986; Berkhout, 1982; Weglein et al., 1997; ten Kroode, 2002; Malcolm et al., 2009).

Until now we have considered events occurring on a single reflector in Kirchhoff modelling eq. (A-4). Suppose the subsurface reflectors are in a configuration where they are separable almost everywhere, i.e. not too dense. The idea behind the linearity approximation on reflectors is to consider a Kirchhoff modelling equation like eq. (A-4) for each reflector and to sum them, in order to account for the contributions of all reflectors. We add subscript k in eq. (A-4) to make explicit that it concerns reflector k:

obviously to $P_{ref,k}^{(ij)}$ and $P_{ref,k}^{(ij)}$, but also to $\theta_k^{(ij)}$ and $G_{inc,k}^{(i)}$ because they describe results of propagation in the medium above S_k excluding S_k, thus different propagation results when different reflectors k are considered. We obtain

$$P_{ref}(\mathbf{r}_s, \mathbf{r}_r, \omega) = \sum_{k\geq 1} \sum_{i,j\geq 1} P_{ref,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega)$$

$$P_{ref,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \omega) = \int_{S_k} d\mathbf{r} R(\mathbf{r}, \theta_k^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r)) \frac{2\cos(\theta_k^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r))}{c(\mathbf{r})} L_{inc,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \mathbf{r}, \omega)$$

$$L_{inc,k}^{(ij)}(\mathbf{r}_s, \mathbf{r}_r, \mathbf{r}, \omega) = i\omega S(\omega) G_{inc,k}^{(i)}(\mathbf{r}_s, \mathbf{r}, \omega) G_{inc,k}^{(j)}(\mathbf{r}, \mathbf{r}_r, \omega). \quad \text{(A-5)}$$

The Born approximation tells us that first-order scattering effects (such as first-order, or single, reflections and diffractions) can be modelled linearly regarding the wavefield if the velocity perturbation is not too strong. Applied to Kirchhoff modelling (based on the reflection coefficient and not on velocity perturbations), this linearity implies that each recorded single reflection event $P_{ref,k}^{(ij)}$ can be associated with one Kirchhoff modelling equation of the form (A-5) if the reflection coefficients are not too large. The total reflected wavefield is obtained by “summing" the $P_{ref,k}^{(ij)}$ (here over the reflectors and the traveltime branches). The traditional linearity approximation on reflectors (Bleistein et al., 2001) considers only the $P_{ref,k}^{(ij)}$ contributions to P_{ref} related to direct source and receiver travel-time branches, i.e., $i \in [1, N(\mathbf{r}_s)]$ and $j \in [1, N(\mathbf{r}_r)]$. Consequently, $G_{inc,k}^{(i)}$ and $\theta_k^{(ij)}$ can be considered as independent of reflectors k (the presence of reflectors above a subsurface position does not condition directly the number of direct waves reaching the subsurface position). We denote them by $G_{inc}^{(i)}$ and $\theta^{(ij)}$ and finally obtain eq. (3).

APPENDIX B

TIME-DEPTH EQUIVALENCY

Time-depth equivalency others things implies that the residual wavelet $f(t)$ present in Kirchhoff-inverted reflectivity maps in the direction perpendicular to the reflectors. In other terms $\delta_{bl}(g_k(\mathbf{r}))$ in eq. (8) represents a band-limited delta distribution that “peaks" in the direction perpendicular to the interface k (Bleistein et al., 2001; Bleistein, 1987). To demonstrate that, we perform a 1^{st}-order Taylor expansion of $g_k(\mathbf{r})$ around the orthogonal projection of $\mathbf{r}$ on interface k, denoted by $\mathbf{r}_k(\mathbf{r})$

$$\begin{aligned}\mathbf{n}(\mathbf{r}) &= g_k(\mathbf{r}_k(\mathbf{r})) + \nabla_{\mathbf{r}_k} g_k(\mathbf{r}_k(\mathbf{r})).(\mathbf{r}-\mathbf{r}_k(\mathbf{r})) + o(|\mathbf{r}-\mathbf{r}_k(\mathbf{r})|^2)\\ &= \mathbf{n}(\mathbf{r}_k(\mathbf{r})).(\mathbf{r}-\mathbf{r}_k(\mathbf{r}))|\nabla_{\mathbf{r}_k} g_k(\mathbf{r}_k(\mathbf{r}))| + o(|\mathbf{r}-\mathbf{r}_k(\mathbf{r})|^2), \quad \text{(B-1)}\end{aligned}$$

where $\mathbf{n}(\mathbf{r}_k(\mathbf{r})) = \frac{\nabla_{\mathbf{r}_k} g_k(\mathbf{r}_k(\mathbf{r}))}{|\nabla_{\mathbf{r}_k} g_k(\mathbf{r}_k(\mathbf{r}))|}$ is the unit vector normal to the interface k

at position $\mathbf{r}_k(\mathbf{r})$ that points “downward". The remainder term is negligible within 0-g.o., i.e. when the interface curvature (the 2nd-order derivatives of g_k) is sufficiently small. Using eq. (B-1), we have

$$\delta_{bl}(g_k(\mathbf{r})) \approx \frac{1}{\left|\nabla_{\mathbf{r}_k} g_k(\mathbf{r}_k(\mathbf{r}))\right|} \delta_{bl}\left(\mathbf{n}(\mathbf{r}_k(\mathbf{r})).(\mathbf{r}-\mathbf{r}_k(\mathbf{r}))\right). \quad \text{(B-2)}$$

Thus the smearing of $\delta_{bl}(g_k(\mathbf{r}))$ due to the residual wavelet F occurs in the direction $\mathbf{n}(\mathbf{r}_k(\mathbf{r}))$.